Functions and relations: supplementary notes

Functions

1. We will follow the definitions and notations in the lecture slides.

$$f : A \to B$$

is a function from the set A to the set B. A function is a *rule* assigning a member of $x \in A$ to exactly one y in B. For example, we often write "let $f(x) = x^2$" on natural numbers $\mathbb{N}$. We are actually giving the rule or recipe for computing the output $x^2 = x * x$ for any input x. Think how you will write a simple programme for x^2.

2. I repeat: a function is a rule that associates to each $x \in A$ exactly one $y \in B$. But several members of A can be associated with the same y.

$$x \begin{smallmatrix} \nearrow y \\ \searrow y' \end{smallmatrix} \text{ :not ok} \qquad \begin{smallmatrix} x \searrow \\ x' \nearrow \end{smallmatrix} y \text{ :ok}$$

Another name for function is *mapping*. If $f(x) = y$ we say x is mapped to y.

3. **How many functions?** Can we count the number of functions from A to B? For finite sets A and B the answer is *yes*. Let us see how to count the number of functions for a simple but important case. Suppose $B = \{0, 1\}$ and $\#(A) = n$ ($\#(A)$ denotes the number of elements in A). For any function $f : A \to B$ it is enough to specify the set of elements A' that are mapped to 0 because the rest are mapped to 1. A' is exactly the subset $f^{\leftarrow}(0)$ of A (see Week3 lecture slide).

4. Let us now look at functions from different perspective. Let $f : A \to B$ be a function. Consider the set $G_f = \{(x, f(x)) | x \in A\}$. Thus G_f is the set of pairs whose first member x is from A and the second member is then $f(x)$. Obviously $G_f \subset A \times B$. It is called the graph of f and is completely determined by f. We can invert the process and define a function as a subset G of $A \times B$ satisfying the following two conditions.

   (a) Let p_1 be the projection (function) $p_1 : A \times B \to A$ such that $p_1((x, y)) = x$. Then $p_1(G) = A$.

   (b) If (x, y) and $(x, y') \in G$ then $y = y'$.

*The first condition says that the set first elements of the pairs in G cover all of A. The second condition is essentially the definition of function. So given G satisfying the two conditions what is the corresponding function, say, g. Answer, $g(x) = p_2(q^{\leftarrow}(x))$ where $q : G \to A$ is the restriction of p_1 to G and p_2 is the projection on to the second member: $p_2((x, y)) = y$.

5. Functions can be defined in strange ways. They are perfectly legitimate but you may have serious problem computing them. Here is a classic example.

   *Consider the collection of all syntactically correct programs in some language like C. Let

   $$\mathcal{P} = \{P | P \text{ is a correct program in C}\}$$

   Here correct means that there are no syntax errors. Let $|P|$ denote the length of the programme P. Define a function

   $$h(P) = \begin{cases} 1 \text{ if } P \text{ eventually stops} \\ 0 \text{ if } P \text{ does not stop} \end{cases}$$

   This is a properly defined function since a programme either stops or does not. A famous result in computer science says that the function h cannot be computed. This means that it is impossible to write a programme which takes any programme P as input and produces $h(P)$.

6. **Example.** We will consider boolean functions. Let

   $$X = \{0, 1\} \text{ and } A = X^n = \underbrace{X \times X \times \cdots \times X}_{n \text{ times}}$$

   Now define functions

   $$\begin{aligned} &\texttt{AND}, \texttt{OR} : X \times X \to X \text{ by} \\ &\texttt{NOT} : X \to X \\ &\texttt{AND}(0,0) = \texttt{AND}(0,1) = \texttt{AND}(1,0) = 0,\ \texttt{AND}(1,1) = 1 \\ &\texttt{OR}(1,0) = \texttt{OR}(0,1) = \texttt{OR}(1,1) = 1,\ \texttt{OR}(0,0) = 0 \\ &\texttt{NOT}(0) = 1,\ \texttt{NOT}(1) = 0 \end{aligned}$$

   These are some of the basic *gates* used in digital circuits. The nice thing is *any* boolean function $f : X^n \to X$ can be implemented by appropriately composing these 3 basic functions. This means that any function that is computable can be computed using a circuit built out of these gates. As a simple example consider the function $f : X^3 \to X$ given by

   $$\begin{aligned} f(0,0,0) = f(0,0,1) = f(0,1,0) = f(1,0,0) = 0 \\ f(0,1,1) = f(1,0,1) = f(1,1,0) = f(1,1,1) = 1 \end{aligned}$$

   Verify that $f(a, b, c) = \texttt{AND}(\texttt{OR}(\texttt{AND}(a, b), c), \texttt{OR}(a, \texttt{AND}(b, c)))$. Please observe how the different functions are composed.

Author: Manas Patra