Exercise sheet 7 – Solutions and Hints

COMP6741: Parameterized and Exact Computation

Serge Gaspers

Semester 2, 2017

Exercise 1. A *Boolean formula in Conjunctive Normal Form (CNF)* is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation). A *HORN* formula is a CNF formula where each clause contains at most one positive literal. For a CNF formula F and an assignment $\tau : S \to \{0,1\}$ to a subset S of its variables, the formula $F[\tau]$ is obtained from F by removing each clause that contains a literal that evaluates to 1 under S, and removing all literals that evaluates to 0 from the remaining clauses.

HORN-Backdoor Detection	
Input:	A CNF formula F and an integer k.
Parameter:	k
Question:	Is there a subset S of the variables of F with $\|S\| \leq k$ such that for each assignment $\tau : S \to \{0,1\}$, the formula $F[\tau]$ is a HORN formula?

Example: $(\neg a \lor b \lor c) \land (b \lor \neg c \lor \neg d) \land (a \lor b \lor \neg e) \land (\neg b \lor c \lor \neg e)$ with $k = 1$ is a Yes-instance, certified by $S = \{b\}$.

- Show that HORN-Backdoor Detection is FPT using the fact that Vertex Cover is FPT.

Hint.

- Show the following: if two distinct positive literals occur in a same clause, then a HORN-backdoor must contain at least one of the corresponding variables.
- Construct a parameterized reduction to Vertex Cover based on these pairwise conflicts.

Exercise 2. Show that Weighted Circuit Satisfiability $\in XP$.

Hint.

- There are n^k assignments of weight k, where n is the number of input gates.

Exercise 3. Recall that a *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Multicolor Clique	
Input:	A graph $G = (V, E)$, an integer k, and a k-coloring of G
Parameter:	k
Question:	Does G have a clique of size k?

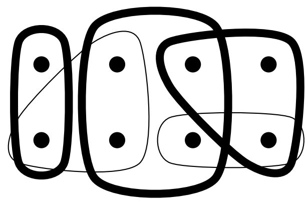
- Show that Multicolor Clique is W[1]-hard.

Solution. The proof is by a parameterized reduction from Clique.
Construction. Let $(G = (V, E), k)$ be an instance for Clique. We construct an instance $(G' = (V', E'), k', f)$ for Multicolor Clique as follows. For each $v \in V$, create k vertices $v(1), \ldots, v(k)$ and add them to V'. For every pair $u(i), v(j) \in V'$ with $i \neq j$, add $u(i)v(j)$ to E' if and only if $uv \in E$. Set $k' := k$. Set $f(v(i)) = i$ for each $v \in V$ and $i \in \{1, \ldots, k\}$.

Equivalence. G has a clique of size k if and only if G' has a clique of size k.
($\Rightarrow$): Let $S = \{s_1, \ldots, s_k\}$ be a clique in G. Then $S' = \{s_1(1), s_2(2), \ldots, s_k(k)\}$ is a clique in G' since $s_i s_j \in E$ implies $s_i(i)s_j(j) \in E'$ in our construction.
($\Leftarrow$): Let S' be a clique of size k in G'. Since for each $i \in \{1, \ldots, k\}$, $\{v_i : v \in V\}$ is an independent set in G', S' contains exactly one vertex from each color class of f. Denote $S' = \{s'_1(1), \ldots, s'_k(k)\}$. Then, $S = \{s_1, \ldots, s_k\}$ is a clique in G.
Parameter. $k' \leq k$.
Running time. The construction can clearly be done in FPT time, and even in polynomial time.

Exercise 4. A *set system* $\mathcal{S}$ is a pair (V, H), where V is a finite set of elements and H is a set of subsets of V. A *set cover* of a set system $\mathcal{S} = (V, H)$ is a subset X of H such that each element of V is contained in at least one of the sets in X, i.e., $\bigcup_{Y \in X} Y = V$.

SET COVER	
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	k
Question:	Does $\mathcal{S}$ have a set cover of cardinality at most k?



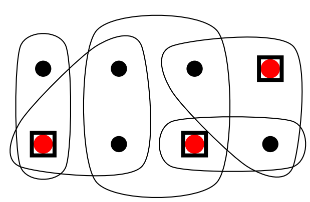
- Show that SET COVER is W[2]-hard.

Hint. Reduce from DOMINATING SET:

- add an element for each vertex and
- add a set for each vertex, containing all the vertices in its closed neighborhood.

Exercise 5. A *hitting set* of a set system $\mathcal{S} = (V, H)$ is a subset X of V such that X contains at least one element of each set in H, i.e., $X \cap Y \neq \emptyset$ for each $Y \in H$.

HITTING SET	
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	k
Question:	Does $\mathcal{S}$ have a hitting set of size at most k?



- Show that HITTING SET is W[2]-hard.

Solution sketch. Reduce from SET COVER. Let $(\mathcal{S} = (V, H), k)$ be an instance for SET COVER. Construct an instance $(\mathcal{S}' = (V', H'), k)$ for HITTING SET:

- $V' := H$
- $H' := \{\{h \in H : v \in h\} : v \in V\}$