Exercise 1. A Boolean formula in Conjunctive Normal Form (CNF) is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation). A HORN formula is a CNF formula where each clause contains at most one positive literal. For a CNF formula $F$ and an assignment $\tau : S \to \{0, 1\}$ to a subset $S$ of its variables, the formula $F[\tau]$ is obtained from $F$ by removing each clause that contains a literal that evaluates to 1 under $S$, and removing all literals that evaluate to 0 from the remaining clauses.

HORN-Backdoor Detection

Input: A CNF formula $F$ and an integer $k$.
Parameter: $k$
Question: Is there a subset $S$ of the variables of $F$ with $|S| \leq k$ such that for each assignment $\tau : S \to \{0, 1\}$, the formula $F[\tau]$ is a HORN formula?

Example: $(\neg a \lor b \lor c) \land (b \lor \neg c \lor \neg d) \land (a \lor b \lor \neg c) \land (\neg b \lor c \lor \neg c)$ with $k = 1$ is a Yes-instance, certified by $S = \{b\}$.

• Show that HORN-Backdoor Detection is FPT using the fact that Vertex Cover is FPT.

Hint.

• Show the following: if two distinct positive literals occur in a same clause, then a HORN-backdoor must contain at least one of the corresponding variables.

• Construct a parameterized reduction to Vertex Cover based on these pairwise conflicts.

Exercise 2. Show that Weighted Circuit Satisfiability $\in NP$.

Hint.

• There are $n^k$ assignments of weight $k$, where $n$ is the number of input gates.

Exercise 3. Recall that a $k$-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to $V$ such that no two adjacent vertices receive the same color.

Multicolor Clique

Input: A graph $G = (V, E)$, an integer $k$, and a $k$-coloring of $G$
Parameter: $k$
Question: Does $G$ have a clique of size $k$?

• Show that Multicolor Clique is W[1]-hard.

Solution. The proof is by a parameterized reduction from CLIQUE.

Construction. Let $(G = (V, E), k)$ be an instance for CLIQUE. We construct an instance $(G' = (V', E'), k', f)$ for Multicolor Clique as follows. For each $v \in V$, create $k$ vertices $v(1), \ldots, v(k)$ and add them to $V'$. For every pair $u(i), v(j) \in V'$ with $i \neq j$, add $u(i)v(j)$ to $E'$ if and only if $uv \in E$. Set $k' := k$. Set $f(v(i)) = i$ for each $v \in V$ and $i \in \{1, \ldots, k\}$. 

1
Equivalence. \( G \) has a clique of size \( k \) if and only if \( G' \) has a clique of size \( k \).

\( \Rightarrow \): Let \( S = \{s_1, \ldots, s_k\} \) be a clique in \( G \). Then \( S' = \{s_1(1), s_2(2), \ldots, s_k(k)\} \) is a clique in \( G' \) since \( s_i s_j \in E \) implies \( s_i(1)s_j(1) \in E' \) in our construction.

\( \Leftarrow \): Let \( S' \) be a clique of size \( k \) in \( G' \). Since for each \( i \in \{1, \ldots, k\} \), \( \{v_i : v \in V\} \) is an independent set in \( G' \), \( S' \) contains exactly one vertex from each color class of \( f \). Denote \( S' = \{s'_1(1), \ldots, s'_k(k)\} \). Then, \( S = \{s_1, \ldots, s_k\} \) is a clique in \( G \).

Parameter. \( k' \leq k \).

Running time. The construction can clearly be done in FPT time, and even in polynomial time.

Exercise 4. A set system \( S \) is a pair \((V, H)\), where \( V \) is a finite set of elements and \( H \) is a set of subsets of \( V \). A set cover of a set system \( S = (V, H) \) is a subset \( X \) of \( H \) such that each element of \( V \) is contained in at least one of the sets in \( X \), i.e., \( \bigcup_{Y \in X} Y = V \).

**Set Cover**

- **Input:** A set system \( S = (V, H) \) and an integer \( k \)
- **Parameter:** \( k \)
- **Question:** Does \( S \) have a set cover of cardinality at most \( k \)?

- Show that Set Cover is W[2]-hard.

**Hint.** Reduce from Dominating Set:

- add an element for each vertex and
- add a set for each vertex, containing all the vertices in its closed neighborhood.

Exercise 5. A hitting set of a set system \( S = (V, H) \) is a subset \( X \) of \( V \) such that \( X \) contains at least one element of each set in \( H \), i.e., \( X \cap Y \neq \emptyset \) for each \( Y \in H \).

**Hitting Set**

- **Input:** A set system \( S = (V, H) \) and an integer \( k \)
- **Parameter:** \( k \)
- **Question:** Does \( S \) have a hitting set of size at most \( k \)?

- Show that Hitting Set is W[2]-hard.

**Solution sketch.** Reduce from Set Cover. Let \((S = (V, H), k)\) be an instance for Set Cover. Construct an instance \((S' = (V', H'), k)\) for Hitting Set:

- \( V' := H \)
- \( H' := \{\{h \in H : v \in h\} : v \in V\} \)