6. Parameterized branching algorithms

COMP6741: Parameterized and Exact Computation

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Outline

1. Running time analysis
2. Feedback Vertex Set
3. Maximum Leaf Spanning Tree
4. Further Reading
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**Recall:** A search tree models the recursive calls of an algorithm. For a $b$-way branching where the parameter $k$ decreases by $a$ at each recursive call, the number of nodes is at most $b^{k/a} \cdot (k/a + 1)$.

If $k/a$ and $b$ are upper bounded by a function of $k$, and the time spent at each node is FPT (typically, polynomial), then we get an FPT running time.
Recall: Measure Based Analysis

For more precise running time upper bounds:

**Lemma 1 (Measure Analysis Lemma)**

Let

- $A$ be a branching algorithm
- $c \geq 0$ be a constant, and
- $\mu(\cdot), \eta(\cdot)$ be two measures for the instances of $A$,

such that on input $I$, $A$ calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O((\eta(I))^c)$, such that

$$
(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \text{ and } \\
2^{\mu(I_1)} + \ldots + 2^{\mu(I_k)} \leq 2^{\mu(I)}.
$$

Then $A$ solves any instance $I$ in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$. 

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A feedback vertex set of a multigraph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G - S$ is acyclic.

**Feedback Vertex Set**

**Input:** Multigraph $G = (V, E)$, integer $k$

**Parameter:** $k$

**Question:** Does $G$ have a feedback vertex set of size at most $k$?
We apply the first applicable\textsuperscript{1} simplification rule.

(Loop)

If $G$ has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

\textsuperscript{1}A simplification rule is applicable if it modifies the instance.
Simplification Rules

We apply the first applicable simplification rule.

(Loop)
If $G$ has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

(Multiedge)
If $E$ contains an edge $uv$ more than twice, remove all but two copies of $uv$.

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If $E$ contains an edge $uv$ more than twice, remove all but two copies of $uv$.

**(Degree-1)**

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$. 
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If $G$ has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

**（Multiedge）**

If $E$ contains an edge $uv$ more than twice, remove all but two copies of $uv$.

**（Degree-1）**

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$.

**（Budget-exceeded）**

If $k < 0$, then return \textbf{No}.

\textsuperscript{1}A simplification rule is \textbf{applicable} if it modifies the instance.
If \( \exists v \in V \) with \( d_G(v) = 2 \), then denote \( N_G(v) = \{u, w\} \) and set \( G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\}) \).
Simplification Rules II

(Degree-2)

If \( \exists v \in V \) with \( d_G(v) = 2 \), then denote \( N_G(v) = \{ u, w \} \) and set 
\[ G \leftarrow G' = (V \setminus \{ v \}, (E \setminus \{ vu, vw \}) \cup \{ uw \}). \]

Lemma 2

(Degree-2) is sound.

Proof.

Suppose \( S \) is a feedback vertex set of \( G \) of size at most \( k \). Let 
\[ S' = \begin{cases} 
S & \text{if } v \notin S \\
(S \setminus \{ v \}) \cup \{ u \} & \text{if } v \in S.
\end{cases} \]

Now, \( |S'| \leq k \) and \( S' \) is a feedback vertex set of \( G' \) since every cycle in \( G' \) corresponds to a cycle in \( G \), with, possibly, the edge \( uw \) replaced by the path \( (u, v, w) \).

Suppose \( S' \) is a feedback vertex set of \( G' \) of size at most \( k \). Then, \( S' \) is also a feedback vertex set of \( G \).
Remaining issues

- A select–discard branching decreases $k$ in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of $k$
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Idea:

- An acyclic graph has average degree $< 2$
- After applying simplification rules, $G$ has average degree $\geq 3$
- The selected feedback vertex set needs to be incident to many edges
- Does a feedback vertex set of size at most $k$ contain at least one vertex among the $f(k)$ vertices of highest degree?
The fvs needs to be incident to many edges

Lemma 3

If \( S \) is a feedback vertex set of \( G = (V, E) \), then

\[
\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1
\]

Proof.

Since \( F = G - S \) is acyclic, \( |E(F)| \leq |V| - |S| - 1 \).
Since every edge in \( E \setminus E(F) \) is incident with a vertex of \( S \), we have

\[
|E| = |E| - |E(F)| + |E(F)| \leq \left( \sum_{v \in S} d_G(v) \right) + (|V| - |S| - 1)
\]

\[
= \left( \sum_{v \in S} (d_G(v) - 1) \right) + |V| - 1.
\]
The fvs needs to contain a high-degree vertex

Lemma 4

Let $G$ be a graph with minimum degree at least 3 and let $H$ denote a set of $3k$ vertices of highest degree in $G$. Every feedback vertex set of $G$ of size at most $k$ contains at least one vertex of $H$. 

Proof. Suppose not. Let $S$ be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

\[
2|E| - |V| = \sum_{v \in V} (d_G(v) - 1) \geq 3 \cdot \left( \sum_{v \in S} (d_G(v) - 1) \right) + \sum_{v \in S} (d_G(v) - 1) \geq 4 \cdot (|E| - |V| + 1).
\]

This contradicts the fact that every vertex of $G$ has degree at least 3.
The fvs needs to contain a high-degree vertex

**Lemma 4**

Let $G$ be a graph with minimum degree at least 3 and let $H$ denote a set of $3k$ vertices of highest degree in $G$.

Every feedback vertex set of $G$ of size at most $k$ contains at least one vertex of $H$.

**Proof.**

Suppose not. Let $S$ be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

\[
2|E| - |V| = \sum_{v \in V} (d_G(v) - 1)
\]

\[
= \sum_{v \in H} (d_G(v) - 1) + \sum_{v \in V \setminus H} (d_G(v) - 1)
\]

\[
\geq 3 \cdot (\sum_{v \in S} (d_G(v) - 1)) + \sum_{v \in S} (d_G(v) - 1)
\]

\[
\geq 4 \cdot (|E| - |V| + 1)
\]

\[
\Leftrightarrow 3|V| \geq 2|E| + 4.
\]

But this contradicts the fact that every vertex of $G$ has degree at least 3.
Algorithm for Feedback Vertex Set

Theorem 5

**Feedback Vertex Set** can be solved in $O^*((3^k)^k)$ time.

**Proof (sketch).**

- Exhaustively apply the simplification rules.
- The branching rule computes $H$ of size $3k$, and branches into subproblems $(G - v, k - 1)$ for each $v \in H$.

Current best: $O^*(3.619^k)$ [Kociumaka, Pilipczuk, 2014]
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A leaf of a tree is a vertex with degree 1. A spanning tree in a graph $G = (V, E)$ is a subgraph of $G$ that is a tree and has $|V|$ vertices.

**Maximum Leaf Spanning Tree**

**Input:** connected graph $G$, integer $k$

**Parameter:** $k$

**Question:** Does $G$ have a spanning tree with at least $k$ leaves?
A $k$-leaf tree in $G$ is a subgraph of $G$ that is a tree with at least $k$ leaves. A $k$-leaf spanning tree in $G$ is a spanning tree in $G$ with at least $k$ leaves.

**Lemma 6**

Let $G = (V, E)$ be a connected graph. $G$ has a $k$-leaf tree $\iff$ $G$ has a $k$-leaf spanning tree.

**Proof.**

$(\Leftarrow)$: trivial

$(\Rightarrow)$: Let $T$ be a $k$-leaf tree in $G$. By induction on $x := |V| - |V(T)|$, we will show that $T$ can be extended to a $k$-leaf spanning tree in $G$.

Base case: $x = 0$ $\checkmark$.

Induction: $x > 0$, and assume the claim is true for all $x' < x$. Choose $uv \in E$ such that $u \in V(T)$ and $v \notin V(T)$. Since $T' := (V(T) \cup \{v\}, E(T) \cup \{uv\})$ has $\geq k$ leaves and $< x$ external vertices, it can be extended to a $k$-leaf spanning tree in $G$ by the induction hypothesis. $\square$
The branching algorithm will check whether $G$ has a $k$-leaf tree.

A tree with $\geq 3$ vertices has at least one internal (= non-leaf) vertex.

“Guess” an internal vertex $r$, i.e., do a $|V|$-way branching fixing an initial internal vertex $r$. 
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In any branch, the algorithm has computed

- $T$ – a tree in $G$
- $I$ – the internal vertices of $T$, with $r \in I$
- $B$ – a subset of the leaves of $T$ where $T$ may be extended: the boundary set
- $L$ – the remaining leaves of $T$
- $X$ – the external vertices $V \setminus V(T)$
The branching algorithm will check whether $G$ has a $k$-leaf tree.

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- $L$ – the remaining leaves of $T$
- $X$ – the external vertices $V \setminus V(T)$

The question is whether $T$ can be extended to a $k$-leaf tree where all the vertices in $L$ are leaves.
Simplification Rules

Apply the first applicable simplification rule:

(Halt-Yes)
If $|L| + |B| \geq k$, then return \textbf{Yes}.

(Halt-No)
If $|B| = 0$, then return \textbf{No}.

(Non-extendable)
If $\exists v \in B$ with $N_G(v) \cap X = \emptyset$, then move $v$ to $L$. 
Lemma 7 (Branching Lemma)

Suppose $u \in B$ and there exists a $k$-leaf tree $T'$ extending $T$ where $u$ is an internal vertex.

Then, there exists a $k$-leaf tree $T''$ extending $(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$. 

Proof. Start from $T'' \leftarrow T'$ and perform the following operation for each $v \in N_G(u) \cap X$.

If $v /\in V(T')$, then add the vertex $v$ and the edge $uv$.

Otherwise, add the edge $uv$, creating a cycle $C$ in $T$ and remove the other edge of $C$ incident to $v$. This does not decrease the number of leaves, since it only increases the number of edges incident to $u$, and $u$ was already internal.
Lemma 7 (Branching Lemma)

Suppose $u \in B$ and there exists a $k$-leaf tree $T'$ extending $T$ where $u$ is an internal vertex.

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Proof.

Start from $T'' \leftarrow T'$ and perform the following operation for each $v \in N_G(u) \cap X$. If $v \notin V(T')$, then add the vertex $v$ and the edge $uv$. Otherwise, add the edge $uv$, creating a cycle $C$ in $T$ and remove the other edge of $C$ incident to $v$. This does not decrease the number of leaves, since it only increases the number of edges incident to $u$, and $u$ was already internal.
Lemma 8 (Follow Path Lemma)

Suppose $u \in B$ and $|N_G(u) \cap X| = 1$. Let $N_G(u) \cap X = \{v\}$.

If there exists a $k$-leaf tree extending $T$ where $u$ is internal, but no $k$-leaf tree extending $T$ where $u$ is a leaf, then there exists a $k$-leaf tree extending $T$ where both $u$ and $v$ are internal.
Lemma 8 (Follow Path Lemma)

Suppose \( u \in B \) and \( |N_G(u) \cap X| = 1 \). Let \( N_G(u) \cap X = \{v\} \).

If there exists a \( k \)-leaf tree extending \( T \) where \( u \) is internal, but no \( k \)-leaf tree extending \( T \) where \( u \) is a leaf, then there exists a \( k \)-leaf tree extending \( T \) where both \( u \) and \( v \) are internal.

Proof.

Suppose not, and let \( T' \) be a \( k \)-leaf tree extending \( T \) where \( u \) is internal and \( v \) is a leaf. But then, \( T - v \) is a \( k \)-leaf tree as well.
Algorithm

- Apply simplification rules
- Select $u \in B$. Branch into
  - $u \in L$
  - $u \in I$. In this case, add $X \cap N_G(u)$ to $B$ (Branching Lemma). In the special case where $|X \cap N_G(u)| = 1$, denote $\{v\} = X \cap N_G(u)$, make $v$ internal, and add $N_G(v) \cap X$ to $B$, continuing the same way until reaching a vertex with at least 2 neighbors in $X$ (Follow Path Lemma).
Apply simplification rules
Select \( u \in B \). Branch into
- \( u \in L \)
- \( u \in I \). In this case, add \( X \cap N_G(u) \) to \( B \) (Branching Lemma). In the special case where \( |X \cap N_G(u)| = 1 \), denote \( \{v\} = X \cap N_G(u) \), make \( v \) internal, and add \( N_G(v) \cap X \) to \( B \), continuing the same way until reaching a vertex with at least 2 neighbors in \( X \) (Follow Path Lemma).

In one branch, a vertex moves from \( B \) to \( L \); in the other branch, \( |B| \) increases by at least 1.
Running time analysis

- Measure $\mu := 2k - 2|L| - |B| \geq 0$.
- Branch where $u \in L$:
  - $|B|$ decreases by 1, $|L|$ increases by 1
  - $\mu$ decreases by 1
- Branch where $u \in I$.
  - $u$ moves from $B$ to $I$
  - $\geq 2$ vertices move from $X$ to $B$
  - $\mu$ decreases by at least 1

- Binary search tree
- Height $\leq \mu \leq 2k$
Theorem 9 ([Kneis, Langer, Rossmanith, 2011])

**Maximum Leaf Spanning Tree** can be solved in $O^*(4^k)$ time.

Current best: $O^*(3.72^k)$ [Daligault, Gutin, Kim, Yeo, 2010]
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