1. Introduction

COMP6741: Parameterized and Exact Computation

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Contents

1 Algorithms for NP-hard problems

Central question

$$\text{P vs. NP}$$

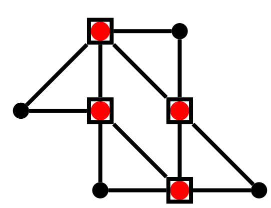
NP-hard problems

- no known polynomial time algorithm for any NP-hard problem
- belief: P $\neq$ NP
- What to do when facing an NP-hard problem?

Example problem: Vertex Cover
A *vertex cover* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

VERTEX COVER	
Input:	Graph G, integer k
Question:	Does G have a vertex cover of size k?

Note: VERTEX COVER is NP-complete.



Coping with NP-hardness

- Approximation algorithms
  - There is an algorithm, which, given an instance (G, k) for Vertex Cover, finds a vertex cover of size at most $2k$ or correctly determines that G has no vertex cover of size k.
- *Exact exponential time algorithms*
  - There is an algorithm solving Vertex Cover in time $O(1.2002^n)$, where $n = |V|$.
- *Fixed parameter algorithms*
  - There is an algorithm solving Vertex Cover in time $O(1.2738^k + kn)$.
- Heuristics
  - Heuristic A finds a smaller vertex cover than Heuristic B on benchmark instances $C_1, \ldots, C_m$.
- Restricting the inputs
  - Vertex Cover can be solved in polynomial time on bipartite graphs, trees, interval graphs, etc.
- Quantum algorithms?
  - Not believed to solve NP-hard problems in polynomial time.

Aims of this course

Design and analyze algorithms for NP-hard problems.

We focus on algorithms that solve NP-hard problems *exactly* and analyze their *worst case running time*.

2 Exponential Time Algorithms

Running times

Worst case running time of an algorithm.

- An algorithm is *polynomial* if $\exists c \in \mathbb{N}$ such that the algorithm solves every instance in time $O(n^c)$, where n is the size of the instance. Also: $n^{O(1)}$ or poly(n).
- *quasi-polynomial*: $2^{O(\log^c n)}$, $c \in O(1)$
- *sub-exponential*: $2^{o(n)}$
- *exponential*: $2^{\text{poly}(n)}$
- *double-exponential*: $2^{2^{\text{poly}(n)}}$

O^*-notation ignores polynomial factors in the input size:

$$O^*(f(n)) \equiv O(f(n) \cdot \mathsf{poly}(n))$$
$$O^*(f(k)) \equiv O(f(k) \cdot \mathsf{poly}(n))$$

Brute-force algorithms for NP-hard problems

Theorem 1. *Every problem in* NP *can be solved in exponential time.*

Proof. Let Π be an arbitrary problem in NP. [Use certificate-based definition of NP] We know that $\exists$ a polynomial p and a polynomial-time verification algorithm V such that:

- for every $x \in \Pi$ (i.e., every YES-instance for Π) $\exists$ string $y \in \{0,1\}^*$, $|y| \le p(|x|)$, such that $V(x,y) = 1$, and
- for every $x \notin \Pi$ (i.e., every NO-instance for Π) and every string $y \in \{0,1\}^*$, $V(x,y) = 0$.

Now, we can prove that there exists an exponential-time algorithm for Π with input x:

- For each string $y \in \{0,1\}^*$ with $|y| \le p(|x|)$, evaluate $V(x,y)$ and return YES if $V(x,y) = 1$.
- Return NO.

Running time: $2^{p(|x|)} \cdot n^{O(1)} \subseteq 2^{O(2 \cdot p(|x|))} = 2^{O(p(|x|))}$, but non-constructive. □

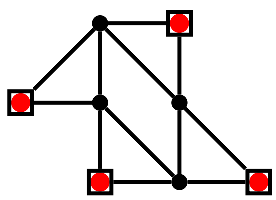
Three main categories for NP-complete problems

- Subset problems
- Permutation problems
- Partition problems

Subset Problem: Independent Set

An *independent set* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that the vertices in S are pairwise non-adjacent in G.

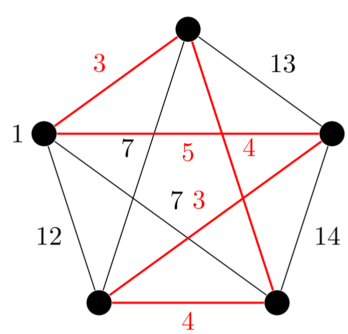
INDEPENDENT SET	
Input:	Graph G, integer k
Question:	Does G have an independent set of size k?



Brute-force: $O^*(2^n)$, where $n = |V(G)|$

Permutation Problem: Traveling Salesman

TRAVELING SALESMAN PROBLEM (TSP)	
Input:	a set of n cities, the distance $d(i,j) \in \mathbb{N}$ between every two cities i and j, integer k
Question:	Is there a permutation of the cities (a *tour*) such that the total distance when traveling from city to city in the specified order, and returning back to the origin, is at most k?

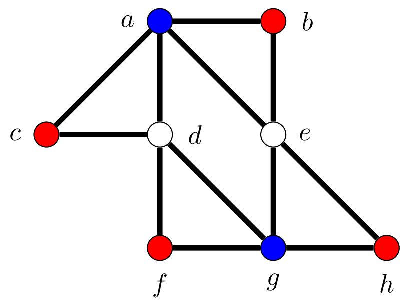


Brute-force: $O^*(n!) \subseteq 2^{O(n \log n)}$

Partition Problem: Coloring

A *k-coloring* of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

COLORING	
Input:	Graph G, integer k
Question:	Does G have a k-coloring?



Brute-force: $O^*(k^n)$, where $n = |V(G)|$

Exponential Time Algorithms

- natural question in Algorithms: design faster (worst-case analysis) algorithms for problems
- might lead to practical algorithms
  - for small instances
    - you don't want to design software where your client/boss can find with better solutions *by hand* than your software
  - subroutines for
    - (sub)exponential time approximation algorithms
    - randomized algorithms with expected polynomial run time

Solve an NP-hard problem

- exhaustive search
  - trivial method
  - try all possible "solutions" (certificates) for a ground set on n elements
  - running times for problems in NP
    - SUBSET PROBLEMS: $O^*(2^n)$
    - PERMUTATION PROBLEMS: $O^*(n!)$
    - PARTITION PROBLEMS: $O^*(c^{n \log n})$
- faster exact algorithms
  - for some problems, it is possible to obtain provably faster algorithms
  - running times $O(1.0892^n), O(1.5086^n), O(1.9977^n)$

Exponential Time Algorithms in Practice

- How large are the instances one can solve in practice?

Available time nb. of operations	1 s 2^{36}	1 min 2^{42}	1 hour 2^{48}	3 days 2^{54}	6 months 2^{60}
n^5	147	337	776	1782	4096
n^{10}	12	18	27	42	64
1.05^n	511	596	681	767	852
1.1^n	261	305	349	392	436
1.5^n	61	71	82	92	102
2^n	36	42	48	54	60
5^n	15	18	20	23	25
$n!$	13	15	16	18	19

Note: Intel Core i7 920 (Quad core) executes between 2^{36} and 2^{37} instructions per second at 2.66 GHz.

> For every polynomial-time algorithm you have, there is an exponential algorithm that I would rather run.
>
> – Alan Perlis (1922-1990, programming languages, 1st recipient of Turing Award)

Hardware vs. Algorithms

- Suppose a 2^n algorithm enables us to solve instances up to size x
- Faster processors
  - processor speed doubles after 18–24 months (Moore's law)
  - can solve instances up to size $x + 1$
- Faster algorithm
  - design an $O^*(2^{n/2}) \subseteq O(1.4143^n)$ time algorithm
  - can solve instances up to size $2 \cdot x$

3 Parameterized Complexity

A story

A computer scientist meets a biologist ... The biologist has performed n experiments. Unfortunately, the data obtained from these experiments has some conflicts. He suspects that a small number k of experiments have gone wrong, and he would like to detect whether removing k experiments can solve all the conflicts.

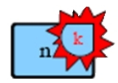
Eliminating conflicts from experiments

$n = 1000$ experiments, $k = 20$ experiments failed

	Running Time	
Theoretical	Number of Instructions	Real
2^n	$1.07 \cdot 10^{301}$	$4.941 \cdot 10^{282}$ years
n^k	10^{60}	$4.611 \cdot 10^{41}$ years
$2^k \cdot n$	$1.05 \cdot 10^9$	0.01526 seconds

Notes:
– We assume that 2^{36} instructions are carried out per second.
– The Big Bang happened roughly $13.5 \cdot 10^9$ years ago.

Goal of Parameterized Complexity
Confine the combinatorial explosion to a parameter k.



For which problem–parameter combinations can we find algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the f is a computable function independent of the input size n?

Examples of Parameters

A Parameterized Problem

Input:	an instance of the problem
Parameter:	a parameter k
Question:	a Yes–No question about the instance and the parameter

- A parameter can be
  - input size (trivial parameterization)
  - solution size
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - etc.

Main Complexity Classes
P: class of problems that can be solved in time $n^{O(1)}$
FPT: class of problems that can be solved in time $f(k) \cdot n^{O(1)}$
W[·]: parameterized intractability classes
XP: class of problems that can be solved in time $f(k) \cdot n^{g(k)}$

$$\mathrm{P} \subseteq \mathrm{FPT} \subseteq \mathrm{W}[1] \subseteq \mathrm{W}[2] \cdots \subseteq \mathrm{W}[P] \subseteq \mathrm{XP}$$

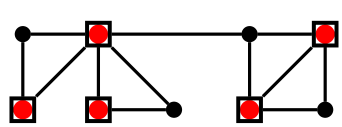
Known: If FPT = W[1], then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in time $2^{o(n)}$.

3.1 FPT Algorithm for Vertex Cover

Vertex Cover

VERTEX COVER (VC)

Input:	A graph $G = (V, E)$ on n vertices, an integer k
Parameter:	k
Question:	Is there a set of vertices $C \subseteq V$ of size at most k such that every edge has at least one endpoint in C?



Algorithm vc1(G, k);

if $E = \emptyset$ **then** `// all edges are covered`
  return Yes
else if $k = 0$ **then** `// we cannot select any vertex`
  return No
else
  Select an edge $uv \in E$;
  return $\text{vc1}(G - u, k - 1) \vee \text{vc1}(G - v, k - 1)$

3.2 Algorithms for Vertex Cover

Brute Force Algorithms

- $2^n \cdot n^{O(1)}$ not FPT
- $n^k \cdot n^{O(1)}$ not FPT

An FPT Algorithm

Running Time Analysis

- Let us look at an arbitrary execution of the algorithm.
- Recursive calls form a *search tree* T
  - with depth $\leq k$
  - where each node has ≤ 2 children
- $\Rightarrow T$ has $\leq 2^k$ leafs and $\leq 2^k - 1$ internal nodes
- at each node the algorithm spends time $n^{O(1)}$
- The running time is $O^*(2^k)$

A faster FPT Algorithm

Algorithm vc2(G, k);

if $E = \emptyset$ **then** `// all edges are covered`
  return Yes
else if $k = 0$ **then** `// we used too many vertices`
  return No
else if $\Delta(G) \leq 2$ **then** `//` G `has maximum degree` ≤ 2
  Solve the problem in polynomial time;
else
  Select a vertex v of maximum degree;
  return $\text{vc2}(G - v, k - 1) \vee \text{vc2}(G - N[v], k - d(v))$

Exercise
Show that VC can be solved in polynomial time for graphs of maxmium degree at most 2.

Solution Idea

Show that VC can be solved in polynomial time for graphs of maximum degree at most 2.

- A graph of maximum degree at most 2 is a disjoint union of paths and cycles
- Denote by $vc_{opt}(G)$ the size of a smallest vertex cover of G

Lemma 2. *For a path P_k on $k \geq 1$ vertices, $vc_{opt}(P_k) = \lceil (k-1)/2 \rceil$.*

Proof sketch. By induction on k. Base cases: check for $k = 1$ and $k = 2$. Induction: Let $k \geq 3$ and assume the lemma is true for $P_{k'}$ for all k', $1 \leq k' < k$. One can prove that $vc_{opt}(P_k) = 1 + vc_{opt}(P_{k-2}) = 1 + \lceil (k-3)/2 \rceil = \lceil (k-1)/2 \rceil$. □

- For a cycle C_k on $k \geq 3$ vertices, $vc_{opt}(C_k) = 1 + vc_{opt}(P_{k-1}) = \lceil k/2 \rceil$.

Running time analysis of vc2

$$\begin{aligned} T(k) &\leq T(k-1) + T(k-3) \\ x^k &\leq x^{k-1} + x^{k-3} \\ x^3 - x^2 - 1 &= 0 \end{aligned}$$

- Positive real root of this equation: $x \approx 1.4655...$
- Running time: $1.4656^k \cdot n^{O(1)}$

4 Further Reading

Further Reading

- Exponential-time algorithms
  - Chapter 1, *Introduction* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.
  - Gerhard J. Woeginger: Exact Algorithms for NP-Hard Problems: A Survey. Combinatorial Optimization 2001: 185-208.
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