3. Branching Algorithms

COMP6741: Parameterized and Exact Computation

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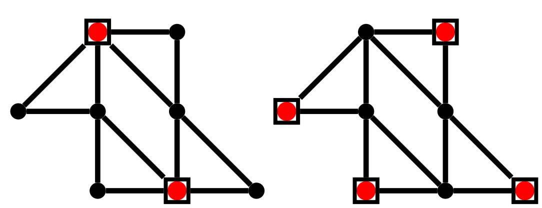
Semester 2, 2015

Contents

1 Introduction

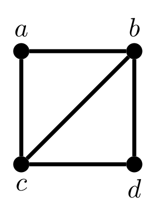
Recall: Maximal Independent Sets

- A vertex set $S \subseteq V$ of a graph $G = (V, E)$ is an *independent set* in G if there is no edge $uv \in E$ with $u, v \in S$.
- An independent set is *maximal* if it is not a subset of any other independent set.
- Examples:



Enumeration problem: Enumerate all maximal independent sets

Enum-MIS	
Input:	graph G
Output:	all maximal independent sets of G



Maximal independent sets: $\{a,d\}, \{b\}, \{c\}$

Note: Let v be a vertex of a graph G. Every maximal independent set contains a vertex from $N_G[v]$.

Branching Algorithm for Enum-MIS

Algorithm ***enum-mis***(G, I)
Input : A graph $G = (V, E)$, an independent set I of G.
Output: All maximal independent sets of G that are supersets of I.

$G' \leftarrow G - N_G[I]$
if $V(G') = \emptyset$ **then** // G' has no vertex
  Output I
else
  Select $v \in V(G')$ such that $d_{G'}(v) = \delta(G')$ // v has min degree in G'
  Run enum-mis$(G, I \cup \{u\})$ for each $u \in N_{G'}[v]$

Running Time Analysis

Define $L(n)$ = largest number of leaves in any search tree of **enum-mis** for an instance with $|V(G')| \leq n$.

Note: $L(n)$ is non-decreasing.

Suppose $d_{G'}(v) = d$ generates a maximum number of leaves. Then,

$$L(n) \leq (d+1) \cdot L(n-(d+1)) = O\left((d+1)^{n/(d+1)}\right)$$

For $s > 0$, the function $f(s) = s^{1/s}$ has its maximum value for $s = e$ and for integer s the maximum value of $f(s)$ is when $s = 3$.

Since the height of the search trees is $\leq |V(G')|$, we obtain:

Theorem 1. *Algorithm* ***enum-mis*** *has running time* $O^*(3^{n/3}) \subseteq O(1.4423^n)$, *where* $n = |V|$.

Corollary 2. *A graph on n vertices has $O(3^{n/3})$ maximal independent sets.*

Constraints Based Analysis

Suppose $L(n) = 2^{\alpha \cdot n}$, $\alpha > 0$.

We constrain for each $d \geq 0$, that

$$2^{\alpha \cdot n} \geq (d+1) \cdot 2^{\alpha \cdot (n-(d+1))},$$

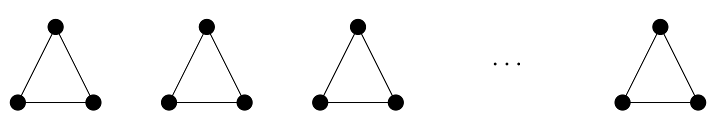
or, equivalently,

$$1 \geq (d+1) \cdot 2^{\alpha \cdot (-(d+1))},$$

and, since we would like to prove a small running time bound, we **minimize** α subject to these constraints.

This amounts to solving a convex program, which gives $\alpha = (1/3) \cdot \log_2 3$ and $L(n) = 2^{(n/3) \cdot \log_2 3} = 3^{n/3}$.

Running Time Lower Bound



Theorem 3. *There is an infinite family of graphs with $\Omega(3^{n/3})$ maximal independent sets.*

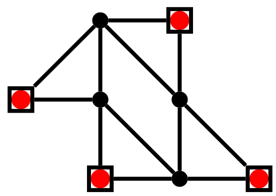
Branching Algorithm

- *Selection*: Select a local configuration of the problem instance
- *Recursion*: Recursively solve subinstances
- *Combination*: Compute an optimal solution of the instance based on the optimal solutions of the subinstances

- *Simplification* rule: 1 recursive call
- *Branching* rule: ≥ 2 recursive calls

2 Maximum Independent Set

Maximum Independent Set	
Input:	graph G
Output:	A largest independent set of G.



Exercise

Suppose there exists a $O^*(1.2^n)$ time algorithm, which, given a graph G on n vertices, computes **the size** of a largest independent set of G.

Design an algorithm, which, given a graph G, **finds** a largest independent set of G in time $O^*(1.2^n)$.

Solution Idea

- Compute k, the size of a largest independent set of G
- Find a vertex v belonging to an independent set of size k
  - We can do this by going through each vertex u of G, and checking whether $G - N_G[u]$ has an independent set of size $k-1$
- Recurse on $(G - N_G[v], k-1)$

Branching Algorithm for Maximum Independent Set

2.1 Simple Analysis

Lemma 4 (Simple Analysis Lemma)**.** *Let*

- *A be a branching algorithm*
- *$\alpha > 0$, $c \geq 0$ be constants*

such that on input I, A calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O(|I|^c)$, such that

$$(\forall i : 1 \leq i \leq k) \quad |I_i| \leq |I| - 1, \text{ and} \tag{1}$$

$$2^{\alpha \cdot |I_1|} + \cdots + 2^{\alpha \cdot |I_k|} \leq 2^{\alpha \cdot |I|}. \tag{2}$$

Then A solves any instance I in time $O(|I|^{c+1}) \cdot 2^{\alpha \cdot |I|}$.

Algorithm ***mis***(G)
Input : A graph $G = (V, E)$.
Output: The size of a maximum i.s. of G.

```
if Δ(G) ≤ 2 then                                        // G has max degree ≤ 2
    return the size of a maximum i.s. of G in polynomial time
else if ∃v ∈ V : d(v) = 1 then                          // v has degree 1
    return 1 + mis(G − N[v])
else if G is not connected then
    Let G_1 be a connected component of G
    return mis(G_1) + mis(G − V(G_1))
else
    Select v ∈ V s.t. d(v) = Δ(G)                       // v has max degree
    return max (1 + mis(G − N[v]), mis(G − v))
```

Proof. By induction on $|I|$. W.l.o.g., replace the hypotheses' O statement with a simple inequality, and for the base case assume that the algorithm returns the solution to an empty instance in time $1 \le |I|^{c+1} 2^{\alpha \cdot |I|}$.

Suppose the lemma holds for all instances of size at most $|I| - 1 \ge 0$, then the running time of algorithm A on instance I is

$$
\begin{aligned}
T_A(I) &\le |I|^c + \sum_{i=1}^{k} T_A(I_i) && \text{(by definition)} \\
&\le |I|^c + \sum |I_i|^{c+1} 2^{\alpha \cdot |I_i|} && \text{(by the inductive hypothesis)} \\
&\le |I|^c + (|I| - 1)^{c+1} \sum 2^{\alpha \cdot |I_i|} && \text{(by (1))} \\
&\le |I|^c + (|I| - 1)^{c+1} 2^{\alpha \cdot |I|} && \text{(by (2))} \\
&\le |I|^{c+1} 2^{\alpha \cdot |I|}.
\end{aligned}
$$

The final inequality uses that $\alpha \cdot |I| > 0$ and holds for any $c \ge 0$. □

Simple Analysis for mis

- At each node of the search tree: $O(n^2)$
- G disconnected:

$$(\forall s : 1 \le s \le n - 1) \quad 2^{\alpha \cdot s} + 2^{\alpha \cdot (n-s)} \le 2^{\alpha \cdot n}. \tag{3}$$

  always satisfied by convexity of the function 2^x

- Branch on vertex of degree $d \ge 3$

$$(\forall d : 3 \le d \le n - 1) \quad 2^{\alpha \cdot (n-1)} + 2^{\alpha \cdot (n-1-d)} \le 2^{\alpha n}. \tag{4}$$

  Dividing all these terms by $2^{\alpha n}$, the constraints become

$$2^{-\alpha} + 2^{\alpha \cdot (-1-d)} \le 1. \tag{5}$$

Compute optimum α
The minimum α satisfying the constraints is obtained by solving a convex mathematical program minimizing α subject to the constraints (the constraint for $d = 3$ is sufficient as all other constraints are weaker).

Alternatively, set $x := 2^\alpha$, compute the unique positive real root of each of the *characteristic polynomials*

$$c_d(x) := x^{-1} + x^{-1-d} - 1,$$

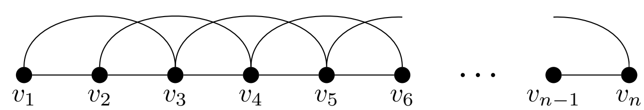
and take the maximum of these roots [Kullmann '99].

d	x	α
3	1.3803	0.4650
4	1.3248	0.4057
5	1.2852	0.3620
6	1.2555	0.3282
7	1.2321	0.3011

Simple Analysis: Result

- use the Simple Analysis Lemma with $c = 2$ and $\alpha = 0.464959$
- running time of Algorithm **mis** upper bounded by $O(n^3) \cdot 2^{0.464959 \cdot n} = O(2^{0.4650 \cdot n})$ or $O(1.3803^n)$

Lower bound



$$T(n) = T(n-5) + T(n-3)$$

- for this graph, P_n^2, the worst case running time is $1.1938\ldots^n \cdot \mathsf{poly}(n)$
- Run time of algo **mis** is $\Omega(1.1938^n)$

Worst-case running time — a mystery

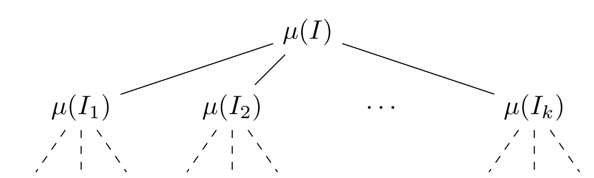
What is the worst-case running time of Algorithm **mis**?

- lower bound $\Omega(1.1938^n)$
- upper bound $O(1.3803^n)$

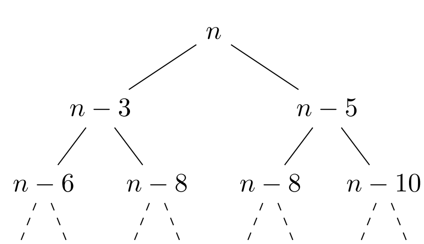
2.2 Search Trees and Branching Numbers

Search Trees

Denote $\mu(I) := \alpha \cdot |I|$.



Example: execution of **mis** on a P_n^2



Branching number: Definition

Consider a constraint

$$2^{\mu(I)-a_1} + \cdots + 2^{\mu(I)-a_k} \leq 2^{\mu(I)}.$$

Its *branching number* is

$$2^{-a_1} + \cdots + 2^{-a_k},$$

and is denoted by

$$(a_1, \ldots, a_k)\,.$$

Clearly, any constraint with branching number at most 1 is satisfied.

Branching numbers: Properties

Dominance For any a_i, b_i such that $a_i \geq b_i$ for all $i,\ 1 \leq i \leq k$,

$$(a_1, \ldots, a_k) \leq (b_1, \ldots, b_k)\,,$$

as $2^{-a_1} + \cdots + 2^{-a_k} \leq 2^{-b_1} + \cdots + 2^{-b_k}$.
In particular, for any $a, b > 0$,

$$\text{either} \quad (a,a) \leq (a,b) \quad \text{or} \quad (b,b) \leq (a,b)\,.$$

Balance If $0 < a \leq b$, then for any ε such that $0 \leq \varepsilon \leq a$,

$$(a,b) \leq (a-\varepsilon, b+\varepsilon)$$

by convexity of 2^x.

Exercises
1. Let A be a branching algorithm, such that, on any input of size at most n its search tree has height at most n and for the number of leaves $L(n)$, we have

$$L(n) \leq 3 \cdot L(n-2)$$

Upper bound the running time of A, assuming it spends only polynomial time at each node of the search tree.

2. Same question, except that

$$L(n) \leq \max \begin{cases} 2 \cdot L(n-3) \\ L(n-2) + L(n-4) \\ 2 \cdot L(n-2) \\ L(n-1) \end{cases}$$

2.3 Measure Based Analysis

- Goal, idea
  - capture more structural changes when branching into subinstances
- Means
  - potential-function method, a.k.a., *Measure & Conquer*
- Example: Algorithm **mis**
  - advantage when degrees of vertices decrease

Multivariate recurrences

- Model running time of **mis** by

$$T(n_1, n_2, \ldots),\ \text{short}\ T\left(\{n_i\}_{i\geq 1}\right),$$

where $n_i := |\{v \in V : d(v) = i\}|$.

- $G - v$: neighbors' degrees decrease
- $G - N[v]$: a vertex in $N^2[v]$ has its degree decreased

Multivariate recurrences (2)

- We obtain the following recurrence where the maximum ranges over all $d \geq 3$, all $p_i, 2 \leq i \leq d$ such that $\sum_{i=2}^d p_i = d$ and all k such that $2 \leq k \leq d$:

$$T\left(\{n_i\}_{i\geq 1}\right) = \max_{d,p_2,\ldots,p_d,k} \begin{cases} T\big(\{n_i - p_i + p_{i+1} - [d = i]\}_{i\geq 1}\big) \\ +T\big(\{n_i - p_i - [d = i] - [k = i] \\ \qquad + [k = i + 1]\}_{i\geq 1}\big) \end{cases} \tag{6}$$

where the Iverson bracket $[F] = \begin{cases} 1 \text{ if } F \text{ true} \\ 0 \text{ otherwise} \end{cases}$

Solve multivariate recurrence

- restrict to max degree 5
- [Eppstein 2004]: there exists a set of weights $w_1, \ldots, w_5 \in \mathbb{R}^+$ such that a solution to (6) is within a polynomial factor of a solution to the corresponding univariate weighted model ($T(\sum_{i=1}^5 \omega_i n_i) = \max \ldots$).

Definition 5. A *measure* μ for a problem P is a function from the set of all instances for P to the set of non negative reals

From recurrences ...

$$\mu(G) := \sum_{i=1}^{5} w_i n_i$$
$$(i \geq 1) \quad w_i \geq 0$$
$$(i \geq 2) \quad w_i \geq w_{i-1}$$
$$(\forall d : 2 \leq d \leq 5) \quad h_d := \min_{2\leq i\leq d} \{w_i - w_{i-1}\}$$

By [Eppstein 2004], there exist weights w_i such that a solution to (6) corresponds to a solution to the following recurrence, where the maximum ranges over all $d, 3 \leq d \leq 5$, and all $p_i, 2 \leq i \leq d$, such that $\sum_{i=2}^d p_i = d$,

$$T(\mu(G)) = \max_{d,p_2,\ldots,p_d,k} \begin{cases} T\left(\mu(G) - w_d - \sum_{i=2}^d p_i \cdot (w_i - w_{i-1})\right) \\ +T\left(\mu(G) - w_d - \sum_{i=2}^d p_i \cdot w_i - h_d\right). \end{cases}$$

... to constraints

$$T(\mu(G)) \geq T\left(\mu(G) - w_d - \sum_{i=2}^d p_i \cdot (w_i - w_{i-1})\right) + T\left(\mu(G) - w_d - \sum_{i=2}^d p_i \cdot w_i - h_d\right)$$

for all $d, 3 \leq d \leq 5$, and all $p_i, 2 \leq i \leq d$, such that $\sum_{i=2}^d p_i = d$.

Measure Based Analysis

Lemma 6 (Measure Analysis Lemma). *Let*

- *A be a branching algorithm*
- *$c \geq 0$ be a constant, and*
- *$\mu(\cdot), \eta(\cdot)$ be two measures for the instances of A,*

such that on input I, A calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O(|I|^c)$, such that

$$(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \text{ and} \tag{7}$$

$$2^{\mu(I_1)} + \ldots + 2^{\mu(I_k)} \leq 2^{\mu(I)}. \tag{8}$$

Then A solves any instance I in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$.

Applying the lemma

$$\begin{aligned} w_i &\geq 0 \\ w_i &\geq w_{i-1} \\ 2^{\mu(G)} &\geq 2^{\mu(G) - w_d - \sum_{i=2}^{d} p_i \cdot (w_i - w_{i-1})} + 2^{\mu(G) - w_d - \sum_{i=2}^{d} p_i \cdot w_i - h_d} \\ &\Leftrightarrow \\ 1 &\geq 2^{-w_d - \sum_{i=2}^{d} p_i \cdot (w_i - w_{i-1})} + 2^{-w_d - \sum_{i=2}^{d} p_i \cdot w_i - h_d} \end{aligned}$$

i	w_i	h_i
1	0	0
2	0.25	0.25
3	0.35	0.10
4	0.38	0.03
5	0.40	0.02

These values for w_i satisfy all the constraints and $\mu(G) \leq 2n/5$ for any graph of max degree ≤ 5. Taking $c = 2$ and $\eta(G) = n$, the Measure Analysis Lemma shows that **mis** has run time $O(n^3)2^{2n/5} = O(1.3196^n)$ on graphs of max degree ≤ 5.

2.4 Optimizing the measure

Compute optimal weights

- By convex programming [Gaspers, Sorkin 2009]

All constraints are already convex, except conditions for h_d

$$(\forall d : 2 \leq d \leq 5) \quad h_d := \min_{2 \leq i \leq d} \{w_i - w_{i-1}\}$$

$$\Downarrow$$

$$(\forall i, d : 2 \leq i \leq d \leq 5) \quad h_d \leq w_i - w_{i-1}.$$

Use existing convex programming solvers to find optimum weights.

convex program in AMPL

```
param maxd integer >= 3;
set DEGREES := 0..maxd;
var W {DEGREES} >= 0;  # weight for vertices according to their degrees
var g {DEGREES} >= 0;  # weight for degree reductions from deg i
var h {DEGREES} >= 0;  # weight for degree reductions from deg \le i
var Wmax;              # maximum weight of W[d]

minimize Obj: Wmax;    # minimize the maximum weight

subject to MaxWeight {d in DEGREES}:
  Wmax >= W[d];
subject to gNotation {d in DEGREES : 2 <= d}:
  g[d] <= W[d]-W[d-1];
subject to hNotation {d in DEGREES, i in DEGREES : 2 <= i <= d}:
  h[d] <= W[i]-W[i-1];
subject to Deg3 {p2 in 0..3, p3 in 0..3 : p2+p3=3}:
  2^(-W[3] -p2*g[2] -p3*g[3]) + 2^(-W[3] -p2*W[2] -p3*W[3] -h[3]) <=1;
subject to Deg4 {p2 in 0..4, p3 in 0..4, p4 in 0..4 : p2+p3+p4=4}:
  2^(-W[4] - p2*g[2] - p3*g[3] - p4*g[4])
+ 2^(-W[4] - p2*W[2] - p3*W[3] - p4*W[4] - h[4]) <=1;
subject to Deg5 {p2 in 0..5, p3 in 0..5, p4 in 0..5, p5 in 0..5 :
                 p2+p3+p4+p5=5}:
  2^(-W[5] - p2*g[2] - p3*g[3] - p4*g[4] - p5*g[5])
+ 2^(-W[5] - p2*W[2] - p3*W[3] - p4*W[4] - p5*W[5] - h[5]) <=1;
```

Optimal weights

i	w_i	h_i
1	0	0
2	0.206018	0.206018
3	0.324109	0.118091
4	0.356007	0.031898
5	0.358044	0.002037

- use the Measure Analysis Lemma with $\mu(G) = \sum_{i=1}^{5} w_i n_i \le 0.358044 \cdot n$, $c = 2$, and $\eta(G) = n$
- **mis** has running time $O(n^3)2^{0.358044 \cdot n} = O(1.2817^n)$

2.5 Exponential Time Subroutines

Lemma 7 (Combine Analysis Lemma)**.** *Let*

- *A be a branching algorithm and B be an algorithm,*
- *$c \ge 0$ be a constant, and*
- *$\mu(\cdot), \mu'(\cdot), \eta(\cdot)$ be three measures for the instances of A and B,*

such that $\mu'(I) \le \mu(I)$ for all instances I, and on input I, A either solves I by invoking B with running time $O(\eta(I)^{c+1}) \cdot 2^{\mu'(I)}$, or calls itself recursively on instances $I_1, \ldots, I_k$, but, besides the recursive calls, uses time $O(|I|^c)$, such that

$$(\forall i) \quad \eta(I_i) \le \eta(I) - 1, \text{ and} \tag{9}$$

$$2^{\mu(I_1)} + \ldots + 2^{\mu(I_k)} \le 2^{\mu(I)}. \tag{10}$$

Then A solves any instance I in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$.

Algorithm mis on general graphs

- use the Combine Analysis Lemma with $A = B =$ **mis**, $c = 2$, $\mu(G) = 0.35805n$, $\mu'(G) = \sum_{i=1}^{5} w_i n_i$, and $\eta(G) = n$
- for every instance G, $\mu'(G) \le \mu(G)$ because $\forall i, w_i \le 0.35805$
- for each $d \ge 6$,

$$(0.35805, (d+1) \cdot 0.35805) \le 1$$

- Thus, Algorithm **mis** has running time $O(1.2817^n)$ for graphs of arbitrary degrees

2.6 Structures that arise rarely

Rare Configurations

- Branching on a local configuration C does not influence overall running time if C is selected only a constant number of times on the path from the root to a leaf of any search tree corresponding to the execution of the algorithm
- Can be proved formally by using measure

$$\mu'(I) := \begin{cases} \mu(I) + c & \text{if } C \text{ may be selected in the current subtree} \\ \mu(I) & \text{otherwise.} \end{cases}$$

Avoid branching on regular instances in mis

else
  Select $v \in V$ such that
    (1) v has maximum degree, and
    (2) among all vertices satisfying (1), v has a neighbor of
      minimum degree
  return $\max\left(1 + \mathbf{mis}(G - N[v]), \mathbf{mis}(G - v)\right)$

New measure:

$$\mu'(G) = \mu(G) + \sum_{d=3}^{5} [G \text{ has a } d\text{-regular subgraph}] \cdot C_d$$

where $C_d, 3 \le d \le 5$, are constants.

Resulting Branching numbers

For each $d, 3 \le d \le 5$ and all $p_i, 2 \le i \le d$ such that $\sum_{i=2}^{d} p_i = d$ *and* $p_d \neq d$,

$$\left(w_d + \sum_{i=2}^{d} p_i \cdot (w_i - w_{i-1}), w_d + \sum_{i=2}^{d} p_i \cdot w_i + h_d\right).$$

All these branching numbers are at most 1 with the optimal set of weights

Result

i	w_i	h_i
1	0	0
2	0.207137	0.207137
3	0.322203	0.115066
4	0.343587	0.021384
5	0.347974	0.004387

Thus, the modified Algorithm **mis** has running time $O(2^{0.3480 \cdot n}) = O(1.2728^n)$.

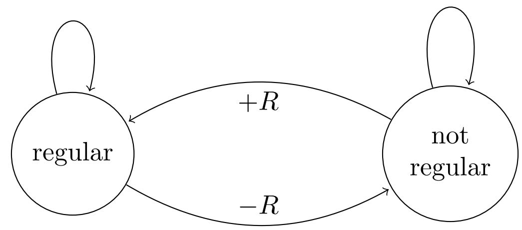
2.7 State Based Measures

State based measures

- "bad" branching always followed by "good" branchings
- *amortize* over branching numbers

$$\mu'(I) := \mu(I) + \Psi(I),$$

where $\Psi : \mathcal{I} \to \mathbb{R}^+$ depends on global properties of the instance.



3 Exercise on Max 2-CSP

MAX 2-CSP

Input: A graph $G = (V, E)$ and a set S of *score functions* containing

- a score function $s_e : \{0,1\}^2 \to \mathbb{N}_0$ for each edge $e \in E$,
- a score function $s_v : \{0,1\} \to \mathbb{N}_0$ for each vertex $v \in V$, and
- a score "function" $s_\emptyset : \{0,1\}^0 \to \mathbb{N}_0$ (which takes no arguments and is just a constant convenient for bookkeeping).

Output: The maximum score $s(\phi)$ of an assignment $\phi : V \to \{0,1\}$:

$$s(\phi) := s_\emptyset + \sum_{v\in V} s_v(\phi(v)) + \sum_{uv\in E} s_{uv}(\phi(u), \phi(v)).$$

1. Design simplification rules for vertices of degree ≤ 2.
2. Using the simple analysis, design and analyze an $O^*(2^{m/4})$ time algorithm, where $m = |E|$.
3. Use the measure $\mu := w_e \cdot m - \left(\sum_{v\in V} w_{d_G(v)}\right)$ to improve the analysis to $O^*(2^{m/5})$.

Solution sketch

Simplification rules

S0 If there is a vertex y with $d(y) = 0$, then set $s_\emptyset = s_\emptyset + \max_{C\in\{0,1\}} s_y(C)$ and delete y from G.

S1 If there is a vertex y with $d(y) = 1$, then denote $N(y) = \{x\}$ and replace the instance with (G', S') where $G' = (V', E') = G - y$ and S' is the restriction of S to V' and E' except that for all $C \in \{0,1\}$ we set

$$s'_x(C) = s_x(C) + \max_{D\in\{0,1\}} \{s_{xy}(C, D) + s_y(D)\}.$$

S2 If there is a vertex y with $d(y) = 2$, then denote $N(y) = \{x, z\}$ and replace the instance with (G', S') where $G' = (V', E') = (V - y, (E \setminus \{xy, yz\}) \cup \{xz\})$ and S' is the restriction of S to V' and E', except that for $C, D \in \{0,1\}$ we set

$$s'_{xz}(C, D) = s_{xz}(C, D) + \max_{F\in\{0,1\}} \{s_{xy}(C, F) + s_{yz}(F, D) + s_y(F)\}$$

if there was already an edge xz, discarding the first term $s_{xz}(C, D)$ if there was not.

Branching rules

B Let y be a vertex of maximum degree. There is one subinstance (G', s^C) for each color $C \in \{0,1\}$, where $G' = (V', E') = G - y$ and s^C is the restriction of s to V' and E', except that we set

$$(s^C)_\emptyset = s_\emptyset + s_y(C),$$

and, for every neighbor x of y and every $D \in \{0,1\}$,

$$(s^C)_x(D) = s_x(D) + s_{xy}(D, C).$$

- Branching on a vertex of degree ≥ 4 removes ≥ 4 edges from both subinstances
- Branching on a vertex of degree 3 removes ≥ 6 edges from both subinstances since G is 3-regular.

The recurrence $T(m) \leq 2 \cdot T(m-4)$ solves to $2^{m/4}$
Using the measure

$$\mu := w_e \cdot m + \left(\sum_{v \in V} w_{d_G(v)} \right)$$

we constrain that

$$\begin{aligned}
w_d &\leq 0 && \text{for all } d \geq 0 \text{ to ensure that } \mu \leq w_e m \\
d \cdot w_e/2 + w_d &\geq 0 && \text{for all } d \geq 0 \text{ to ensure that } \mu(G) \geq 0 \\
-w_0 &\leq 0 && \text{constraint for S0} \\
-w_2 - w_e &\leq 0 && \text{constraint for S2}
\end{aligned}$$

$$1 - w_d - d \cdot w_e - d \cdot (w_j - w_{j-1}) \leq 0$$

for all $d, j \geq 3$.

Using $w_e = 0.2$, $w_0 = 0$, $w_1 = -0.05$, $w_2 = -0.2$, $w_3 = -0.05$, and $w_d = 0$ for $d \geq 4$, all constraints are satisfied and $\mu(G) \leq m/5$ for each graph G.

4 Further Reading

- Chapter 2, *Branching* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.
- Chapter 6, *Measure & Conquer* in Fedor V. Fomin and Dieter Kratsch. Exact Exponential Algorithms. Springer, 2010.
- Chapter 2, *Branching Algorithms* in Serge Gaspers. Exponential Time Algorithms: Structures, Measures, and Bounds. VDM Verlag Dr. Mueller, 2010.