8. Parameterized intractability: the W-hierarchy
COMP6741: Parameterized and Exact Computation

Serge Gaspers[1 2]

[1]School of Computer Science and Engineering, UNSW Australia
[2]Optimisation Resarch Group, NICTA

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Outline

1. Reminder: Polynomial Time Reductions and NP-completeness
2. Parameterized Complexity Theory
   - Parameterized reductions
   - Parameterized complexity classes
3. Case studies
4. Further Reading

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Polynomial-time reduction

Definition 1

A polynomial-time reduction from a decision problem Π_1 to a decision problem Π_2 is a polynomial-time algorithm, which, for any instance of Π_1 produces an equivalent instance of Π_2.
If there exists a polynomial-time reduction from Π_1 to Π_2, we say that Π_1 is polynomial-time reducible to Π_2 and write $\Pi_1 \leq_{\mathsf{P}} \Pi_2$.

New polynomial-time algorithms via reductions

Lemma 2

If Π_1, Π_2 *are decision problems such that* $\Pi_1 \leq_{\mathsf{P}} \Pi_2$*, then* $\Pi_2 \in P$ *implies* $\Pi_1 \in P$.

NP-completeness

Definition 3 (NP-hard)

A decision problem Π is NP-hard if $\Pi' \leq_P \Pi$ for every $\Pi' \in$ NP.

Definition 4 (NP-complete)

A decision problem Π is NP-complete (in NPC) if

1. $\Pi \in$ NP, and
2. Π is NP-hard.

Proving NP-completeness

Lemma 5

If Π *is a decision problem such that* $\Pi' \leq_{\mathsf{P}} \Pi$ *for some* NP*-hard decision problem* Π'*, then* Π *is* NP*-hard.*
If, in addition, $\Pi \in \mathsf{NP}$*, then* $\Pi \in \mathsf{NPC}$*.*

Proving NP-completeness II

Method to prove that a decision problem Π is NP-complete:

1. Prove $\Pi \in$ NP
2. Prove Π is NP-hard.
   - Select a known NP-hard decision problem Π'.
   - Describe an algorithm that transforms every instance I of Π' to an instance $r(I)$ of Π.
   - Prove that for each instance I of Π', we have that I is a YES-instance of Π' $\Leftrightarrow r(I)$ is a YES-instance of Π.
   - Show that the algorithm runs in polynomial time.

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Main Parameterized Complexity Classes

n: instance size
k: parameter

P: class of problems that can be solved in $n^{O(1)}$ time
FPT: class of parameterized problems that can be solved in $f(k) \cdot n^{O(1)}$ time
W[$\cdot$]: parameterized intractability classes
XP: class of parameterized problems that can be solved in $f(k) \cdot n^{g(k)}$ time
("polynomial when k is a constant")

$$\mathsf{P} \subseteq \mathsf{FPT} \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \cdots \subseteq \mathsf{W}[P] \subseteq \mathsf{XP}$$

Note: We assume that f is computable and non-decreasing.

Polynomial-time reductions for parameterized problems?

A vertex cover in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every edge of G has an endpoint in S.

VERTEX COVER	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have a vertex cover of size k?

An independent set in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that there is no edge $uv \in E$ with $u, v \in S$.

INDEPENDENT SET	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have an independent set of size k?

Polynomial-time reductions for parameterized problems?

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Vertex Cover	
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Question:	Does G have a vertex cover of size k?

An independent set in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that there is no edge $uv \in E$ with $u, v \in S$.

Independent Set	
Input:	Graph G, integer k
Parameter:	k
Question:	Does G have an independent set of size k?

- We know: Independent Set $\leq_{\mathsf{P}}$ Vertex Cover
- However: Vertex Cover $\in$ FPT but Independent Set is not known to be in FPT

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- Issue with polynomial-time reductions: parameter can change arbitrarily

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- We will want the reduction to produce an instance where the parameter is bounded by a function of the original instance

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- Issue with polynomial-time reductions: parameter can change arbitrarily
- We will want the reduction to produce an instance where the parameter is bounded by a function of the original instance
- Also: we can allow the reduction to take FPT time instead of only polynomial time.

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Parameterized reduction

Definition 6

A parameterized reduction from a parameterized decision problem Π_1 to a parameterized decision problem Π_2 is an algorithm, which, for any instance I of Π_1 with parameter k produces an instance I' of Π_2 with parameter k' such that

- I is a YES-instance for $\Pi_1 \Leftrightarrow I'$ is a YES-instance for Π_2,
- there exists a computable function g such that $k' \leq g(k)$, and
- there exists a computable function f such that the running time of the algorithm is $f(k) \cdot |I|^{O(1)}$.

If there exists a parameterized reduction from Π_1 to Π_2, we write $\Pi_1 \leq_{\mathsf{FPT}} \Pi_2$.

Note: We can assume that f and g are non-decreasing.

New FPT algorithms via reductions

Lemma 7

If Π_1, Π_2 are parameterized decision problems such that $\Pi_1 \leq_{\mathsf{FPT}} \Pi_2$, then $\Pi_2 \in \mathsf{FPT}$ implies $\Pi_1 \in \mathsf{FPT}$.

Proof.

Exercise. □

Exercise

A Boolean formula in Conjunctive Normal Form (CNF) is a conjunction (AND) of disjunctions (OR) of literals (a Boolean variable or its negation).
A HORN formula is a CNF formula where each clause contains at most one positive literal.
For a CNF formula F and an assignment $\tau : S \to \{0,1\}$ to a subset S of its variables, the formula $F[\tau]$ is obtained from F by removing each clause that contains a literal that evaluates to 1 under S, and removing all literals that evaluate to 0 from the remaining clauses.

HORN-Backdoor Detection	
Input:	A CNF formula F and an integer k.
Parameter:	k
Question:	Is there a subset S of the variables of F with $\|S\| \leq k$ such that for each assignment $\tau : S \to \{0,1\}$, the formula $F[\tau]$ is a HORN formula?

Example: $(\neg a \vee b \vee c) \wedge (b \vee \neg c \vee \neg d) \wedge (a \vee b \vee \neg e) \wedge (\neg b \vee c \vee \neg e)$ with $k = 1$ is a Yes-instance, certified by $S = \{b\}$.

- Show that HORN-Backdoor Detection is FPT using the fact that Vertex Cover is FPT.

Outline

Boolean Circuits

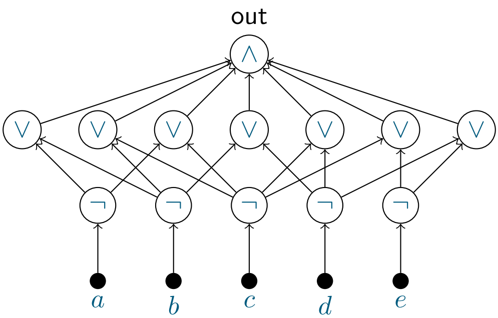
Definition 8

A Boolean circuit is a directed acyclic graph with the nodes labeled as follows:

- every node of in-degree 0 is an input node,
- every node with in-degree 1 is a negation node ($\neg$), and
- every node with in-degree ≥ 2 is either an AND-node ($\wedge$) or an OR-node ($\vee$).

Moreover, exactly one node with out-degree 0 is also labeled the output node.
The depth of the circuit is the maximum length of a directed path from an input node to the output node.
The weft of the circuit is the maximum number of nodes with in-degree ≥ 3 on a directed path from an input node to the output node.

Example



A depth-3, weft-1 Boolean circuit with inputs a, b, c, d, e.

Weighted Circuit Satisfiability

Given an assignment of Boolean values to the input gates, the circuit determines Boolean values at each node in the obvious way.
If the value of the output node is 1 for an input assignment, we say that this assignment satisfies the circuit.
The weight of an assignment is its number of 1s.

Weighted Circuit Satisfiability (WCS)	
Input:	A Boolean circuit C, an integer k
Parameter:	k
Question:	Is there an assignment with weight k that satisfies C?

Exercise: Show that Weighted Circuit Satisfiability $\in$ XP.

WCS for special circuits

Definition 9

The class of circuits $\mathcal{C}_{t,d}$ contains the circuits with weft $\leq t$ and depth $\leq d$.

For any class of circuits $\mathcal{C}$, we can define the following problem.

WCS[$\mathcal{C}$]

Input:	A Boolean circuit $C \in \mathcal{C}$, an integer k
Parameter:	k
Question:	Is there an assignment with weight k that satisfies C?

W classes

Definition 10 (W-hierarchy)

Let $t \in \{1, 2, \dots\}$. A parameterized problem Π is in the parameterized complexity class W[t] if there exists a paramterized reduction from Π to WCS[$\mathcal{C}_{t,d}$] for some constant $d \geq 1$.

Independent Set and Dominating Set

Theorem 11

INDEPENDENT SET $\in$ W[1].

Theorem 12

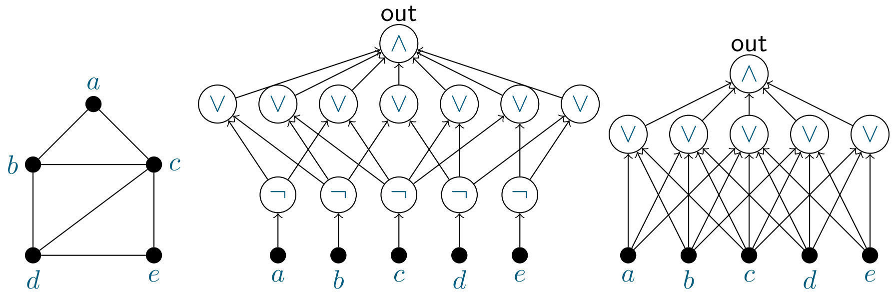
DOMINATING SET $\in$ W[2].

Recall: A dominating set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $N_G[S] = V$.

DOMINATING SET	
Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have a dominating set of size at most k?

“Proof” by picture

Parameterized reductions from Independent Set to WCS[$\mathcal{C}_{1,3}$] and from Dominating Set to WCS[$\mathcal{C}_{2,2}$].



Setting an input node to 1 corresponds to adding the corresponding vertex to the independent set / dominating set.

W-hardness

Definition 13

Let $t \in \{1, 2, \ldots\}$.
A parameterized decision problem Π is W[t]-hard if for every parameterized decision problem Π' in W[t], there is a parameterized reduction from Π' to Π.
Π is W[t]-complete if $\Pi \in$ W[t] and Π is W[t]-hard.

It has been proved that Independent Set is W[1]-hard and Dominating Set is W[2]-hard. Therefore,

Theorem 14

Independent Set *is* W[1]*-complete.*

Theorem 15

Dominating Set *is* W[2]*-complete.*

Proving W-hardness

To show that a parameterized decision problem Π is W[t]-hard:

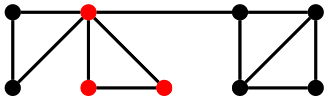
- Select a W[t]-hard problem Π'
- Show that $\Pi' \leq_{\mathsf{FPT}} \Pi$ by designing a parameterized reduction from Π' to Π
  - Design an algorithm, that, for any instance I' of Π' with parameter k', produces an equivalent instance I of Π with parameter k
  - Show that k is upper bounded by a function of k'
  - Show that there exists a function f such that the running time of the algorithm is $f(k') \cdot |I'|^{O(1)}$

Outline

Clique

A clique in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every two vertices from S are adjacent in G.

CLIQUE	
Input:	Graph $G = (V, E)$, integer k
Parameter:	k
Question:	Does G have a clique of size k?



- We will show that CLIQUE is W[1]-hard by a parameterized reduction from INDEPENDENT SET.

Clique is W[1]-hard

Lemma 16

INDEPENDENT SET $\leq_{\mathsf{FPT}}$ CLIQUE.

Proof.

Given any instance $(G = (V, E), k)$ for INDEPENDENT SET, we need to describe an FPT algorithm that constructs an equivalent instance (G', k') for CLIQUE such that $k' \leq g(k)$ for some computable function g.

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Construction. Set $k' \leftarrow k$ and $G' \leftarrow \overline{G} = (V, \{uv : u, v \in V, u \neq v, uv \notin E\})$.

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Equivalence. We need to show that (G, k) is a YES-instance for INDEPENDENT SET if and only if (G', k') is a YES-instance for CLIQUE.

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Equivalence. We need to show that (G, k) is a YES-instance for INDEPENDENT SET if and only if (G', k') is a YES-instance for CLIQUE.

$(\Rightarrow)$: Let S be an independent set of size k in G. For every two vertices $u, v \in S$, we have that $uv \notin E$. Therefore, $uv \in E(\overline{G})$ for every two vertices in S. We conclude that S is a clique of size k in $\overline{G}$.

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$(\Leftarrow)$: Let S be a clique of size k in $\overline{G}$. By a similar argument, S is an independent set of size k in G.

Clique is W[1]-hard

Lemma 16

Independent Set $\leq_{\text{FPT}}$ Clique.

Proof.

Given any instance $(G = (V, E), k)$ for Independent Set, we need to describe an FPT algorithm that constructs an equivalent instance (G', k') for Clique such that $k' \leq g(k)$ for some computable function g.

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$(\Leftarrow)$: Let S be a clique of size k in $\overline{G}$. By a similar argument, S is an independent set of size k in G.

Parameter. $k' \leq k$.

Clique is W[1]-hard

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Given any instance $(G = (V, E), k)$ for INDEPENDENT SET, we need to describe an FPT algorithm that constructs an equivalent instance (G', k') for CLIQUE such that $k' \leq g(k)$ for some computable function g.

Construction. Set $k' \leftarrow k$ and $G' \leftarrow \overline{G} = (V, \{uv : u, v \in V, u \neq v, uv \notin E\})$.

Equivalence. We need to show that (G, k) is a YES-instance for INDEPENDENT SET if and only if (G', k') is a YES-instance for CLIQUE.

$(\Rightarrow)$: Let S be an independent set of size k in G. For every two vertices $u, v \in S$, we have that $uv \notin E$. Therefore, $uv \in E(\overline{G})$ for every two vertices in S. We conclude that S is a clique of size k in $\overline{G}$.

$(\Leftarrow)$: Let S be a clique of size k in $\overline{G}$. By a similar argument, S is an independent set of size k in G.

Parameter. $k' \leq k$.

Running time. The construction can clearly be done in FPT time, and even in polynomial time. □

Clique is W[1]-hard II

Corollary 17

CLIQUE *is* W[1]*-hard*

Exercise

Recall: A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

MULTICOLOR CLIQUE	
Input:	A graph $G = (V, E)$, an integer k, and a k-coloring of G
Parameter:	k
Question:	Does G have a clique of size k?

- Show that MULTICOLOR CLIQUE is W[1]-hard.

Exercise

Recall: A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

Multicolor Clique	
Input:	A graph $G = (V, E)$, an integer k, and a k-coloring of G
Parameter:	k
Question:	Does G have a clique of size k?

- Show that Multicolor Clique is W[1]-hard.

Hint: Reduce from Clique, and create k copies of V, each one being an independent set in G'.

Exercise

Recall: A k-coloring of a graph $G = (V, E)$ is a function $f : V \to \{1, 2, ..., k\}$ assigning colors to V such that no two adjacent vertices receive the same color.

MULTICOLOR CLIQUE	
Input:	A graph $G = (V, E)$, an integer k, and a k-coloring of G
Parameter:	k
Question:	Does G have a clique of size k?

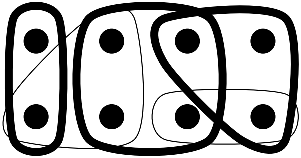
- Show that MULTICOLOR CLIQUE is W[1]-hard.

Hint: Reduce from CLIQUE, and create k copies of V, each one being an independent set in G'. Add edges to enforce constraints that a clique of size k in G' corresponds to a clique of size k in G, and vice-versa.

Exercise

A set system $\mathcal{S}$ is a pair (V, H), where V is a finite set of elements and H is a set of subsets of V.
A set cover of a set system $\mathcal{S} = (V, H)$ is a subset X of H such that each element of V is contained in at least one of the sets in X, i.e., $\bigcup_{Y \in X} Y = V$.

Set Cover	
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	k
Question:	Does $\mathcal{S}$ have a set cover of cardinality at most k?



- Show that Set Cover is W[2]-hard.

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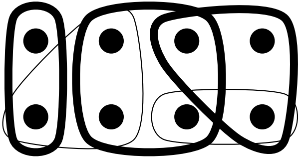


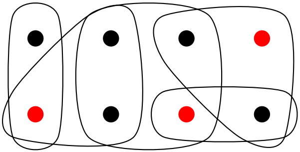
- Show that Set Cover is W[2]-hard.

Hint: Reduce from Dominating Set.

Exercise

A hitting set of a set system $\mathcal{S} = (V, H)$ is a subset X of V such that X contains at least one element of each set in H, i.e., $X \cap Y \neq \emptyset$ for each $Y \in H$.

Hitting Set	
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	k
Question:	Does $\mathcal{S}$ have a hitting set of size at most k?



- Show that Hitting Set is W[2]-hard.

A hitting set of a set system $\mathcal{S} = (V, H)$ is a subset X of V such that X contains at least one element of each set in H, i.e., $X \cap Y \neq \emptyset$ for each $Y \in H$.

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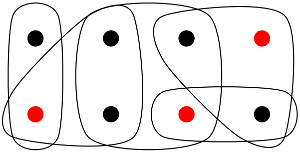
Input:	A set system $\mathcal{S} = (V, H)$ and an integer k
Parameter:	k
Question:	Does $\mathcal{S}$ have a hitting set of size at most k?



- Show that HITTING SET is W[2]-hard.

Hint: Exploit a duality between sets and elements in set covers and hitting sets.

Outline

Further Reading

- Chapter 13, *Fixed-parameter Intractability* in
  Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, MichałPilipczuk, and Saket Saurabh. Parameterized Algorithms. Springer, 2015.
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