Functions and relations 2: supplementary notes

Relations

1. In a sense relations are all we study in mathematics. So let us start with some familiar examples. Consider the natural numbers $\mathbb{N}$. Let us say that two numbers a and b are related if $a < b$. This is a relation between two elements of $\mathbb{N}$, a *binary* relation. There are also ternary relations, relations among 3 objects of a set. Consider, for example, positive integers triples (a, b, c) which can make up the sides of a right-angle triangle. We must have $c^2 = a^2 + b^2$. This defines a relation among the tree numbers.

2. Relations can be defined among members of different sets. See examples in the slides and the textbooks. Formally an n-ary relation on the sets $A_1, A_2, \ldots, A_n$ is a subset of $A_1 \times A_2 \times \cdots \times A_n$. A relation is called unary if $n = 1$, binary if $n = 2$ and ternary if $n = 3$ and so on. Note that some of the sets A_i maybe identical.

3. We will mainly focus on binary relations an a set A. That means we will be looking at subsets of $A \times A$. Let $R \subset A \times A$ be a binary relation. A very important example of binary relation is a graph which will be studied later. We say $a, b \in A$ are related if $(a, b) \in R$. Sometimes this is written as aRb. Let us define some special properties of binary relations.

    (a) The relation R is called *reflexive* if *for all* $a \in A$, $(a, a) \in R$. In words, every member is related to itself. In the opposite direction a relation R is called *irreflexive* if *no* member is related to itself: for any $a \in A$, $(a, a) \notin R$.

    Let $A = \{0, 1, 2\}$ and $R = \{(0, 0), (1, 2), (2, 3)\}$. R is neither reflexive nor irreflexive. The important thing to look out for is that in both definitions must be satisfied for all members.

    (b) A relation is called *symmetric* if for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \in R$. It is called *asymmetric* if for all $a, b \in A$, $(a, b) \in R$ implies $(b, a) \notin R$. A relation may be neither symmetric nor asymmetric. Can you come up with an example? Again the keyword for both properties is 'all'. Observe that in the above definitions we allow $a = b$. So an asymmetric relation must be irreflexive.

    (c) A relation R is called *antisymmetric* if $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$. Note that there are two conditions in the antecedent. If both do not hold then the antecedent is false. When you do propositional logic you will see that by definition the formula $p \Rightarrow q$ is true if p is *false* or q is *true*. The only case where it is false is p is true *and* q is false.

(d) *A relation R is asymmetric if and only if it is antisymmetric *and* irreflexive. Let us prove the 'if' part. Suppose R is antisymmetric *and* irreflexive. Assume it is not asymmetric. Then there exist $a, b \in A$ such that both (a, b) and $(b, a) \in R$. Since R is antisymmetric $a = b$, that is, $(a, a) \in R$. But by hypothesis R is irreflexive. So our assumption must be wrong, that is, R must be asymmetric.

4. A relation R is called transitive if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

   Exercise. A relation R is transitive and irreflexive implies it is asymmetric.

   Proof. Again we prove by contradiction. The idea is to show that if we assume that assertion is false then it leads to a contradiction of the hypothesis. (in this case: R is transitive and irreflexive). So assume that R is *not* asymmetric. Then there must exist pairs $(a, b) \in R$ and $(b, a) \in R$. Then transitivity part of the hypothesis implies that $(a, a) \in R$ which contradicts the irreflexive property in the hypothesis.

5. A relation is called a *partial order* if it is reflexive, antisymmetric and transitive. This is a very important type of relation and so many books have a special symbol $\preceq$ for a generic partial order relation and we write $a \preceq b$ for $(a, b) \in R$. For a partial order $\preceq$ on a set A and for all $a, b \in A$ if it is the case that either $a \preceq b$ or $b \preceq a$ then the relation is called *total*. Let us look at some examples of partial order.

   (a) Divisibility. Let $\mathbb{N}_+$ denote the positive integers. Write $m|n$ if m is a factor of n (m divides n). If we let $R = \{(m, n) \in R : m|n\}$ then R is partial order. Note that it is possible that neither m divides n nor n divides m ((3,5) for example). So two numbers are 'unrelated' with respect to the relation R. That is why the adjective 'partial'.

   (b) Let Σ be an alphabet. Consider the relations *lex* and *lenlex* on Σ^*. Both are total order relations.

   (c) Let $\mathbb{R}$ be the set of real numbers. The standard relation $x \leq y$ if x is less than or equal to y is total order on $\mathbb{R}$.

   (d) Consider the power set pow(A) of A. For $X, Y \in$ pow(A) (they are subsets of A) define $X \preceq Y$ if $X \subseteq Y$.

   (e) Let A be the set of all humans (dead or alive!). For $x, y \in A$ if we define $x \preceq y$ if x is an ancestor of y then $\preceq$ is a partial order assuming that an individual is an ancestor of herself/himself.

6. Try to prove that the relations defined in the examples above are partial orders. You have to show that all the three conditions are satisfied.

7. Let $\preceq$ be a partial order on A. For $a, b \in A$ define $b \succeq a$ if $a \preceq b$. Thus $\succeq$ is the reverse relation of $\preceq$. Note that $\succeq$ is also a partial order. The reflexivity and symmetry properties are easy. Let us look at transitivity. If $c \succeq b$ and $b \succeq a$ then by definition $b \preceq c$ and $a \preceq b$. So by transitivity of $\preceq$, $a \preceq c$. Hence $c \succeq a$ from the definition of $\succeq$.

8. A function is a special kind of relation. See the supplementary notes on functions.

You will see more relations and functions later in the course.
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