Functions and relations 3: equivalence relations

Relations

1. An *equivalence relations* (EV) is by definition a binary relation that is reflexive, symmetric ans transitive. The simplest EV is equality. In fact, equivalence relations are a generalisation of equality relation.

2. Explicitly, suppose R is an equivalence relation on a set A. Then $R \subset A \times A$ and for all $x, y \in A$, $(x, x) \in R$, $(x, y)R$ implies $(y, x) \in R$ and $(x, y) \in R$ *and* $(y, z) \in R$ implies $(x, z) \in R$.

3. An equivalence relation on set A partitions the set into disjoint subsets called equivalence classes. This is a very important concept. Let us dwell on it a bit. Suppose R is an equivalence relation on A. For any $x \in A$ consider the subset $R_x = \{y \in A | (x, y) \in R\}$. That is, R_x consists of all elements which are related to x. R_x is called the *equivalence class* of x. First, $x \in R_x$ (why?). So the sets R_x as we run through different $x \in A$ are not empty. Now, suppose x, y are distinct elements then either $R_x = R_y$ or $R_x \cap R_y = \emptyset$. Thus two equivalence classes are either disjoint or they are identical. What is ruled out is that they are not same but have some common members. You have seen the proof in the lectures. You need both symmetry and transitivity to prove it.

   The set A is the union of disjoint non-empty equivalence classes. That is,

   $$A = R_{x_1} \cup R_{x_2} \cup R_{x_3} \cup \cdots$$

   where the sets R_{x_i} and R_{x_j} are disjoint if $i \neq j$.

4. Writing a set as above, as a union of non-empty disjoint subsets, is called a partition of the set. Any partition of a set $A = A_1 \cup A_2 \cup A_3 \cup \cdots$ defines an equivalence relation Q: $(x, y) \in Q$ if both x, y belong to the same subset, say, A_i in the partition. The A_i are equivalence classes for this relation. Note there may be infinite number of equivalence classes. There is a one-to-one correspondence between equivalence relations and partitions on a set.

5. (a) Let us look at some examples of equivalence relations. In plane geometry, consider the set of lines. Let us define a relation Pl between lines by saying $(l_1, l_2) \in Pl$ if l_1 is parallel to l_2. You can easily check that this is an equivalence relation.

(b) Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. If (m_1, c_1) and (m_2, c_2) are two elements of $\mathbb{R}^2$ then define a relation Q on $\mathbb{R}^2$ by $((m_1, c_1), (m_2, c_2)) \in Q$ if $m_1 = m_2$. Again you can verify that Q is indeed an equivalence relation. What is the connection between this example and the previous one? Recall that a line can be characterised by a pair of (real) numbers (m, c) where m is the slope and c is the and c is the intercept ($y = mx + c$ is the equation of the line). Two lines are parallel if and only if they have the same slope.

(c) Let $R \subset \mathbb{N} \times \mathbb{N}$ be the relation defined by $(x, y) \in R$ if $x + y$ is even. R is an equivalence relation. Reflexivity and symmetry are easy. $x + y$ is even if both either both x, y are even or both odd. So transitivity is verified (work it out!). What are the equivalence classes? There are two, the first consisting of odd numbers and the second of even numbers.

(d) The preceding example defines an equivalence relation that divides natural numbers into two subsets depending on the remainder when divided by 2. We can generalise to any number > 1 instead of 2. Fix a number n. Recall that for a number m the possible remainders when divided by n are $r = 0, 1, \ldots, n-1$ ($m = qn + r$). Each value of the remainder defines an equivalence class. These have a special name. Thus $r \bmod n$ is the equivalence class consisting of all numbers that have remainder r. For example, $2 \bmod 3 = \{2, 5, 8, 11, \cdots, \}$.

(e) Let us consider a "non-mathematical" example. Let A be the students in this course. For any two students x, y, $(x, y) \in LN \subset A \times A$ if they have last names beginning with the same letter. This is an equivalence relations. How many equivalence classes are there and what are they?

(f) All the examples above can be considered as special cases of the following example. Let $f : A \to B$ be a function from A to B. Define a relation $R_f \subset A \times A$ on A as follows. $(x, y) \in R_f$ if and only if $f(x) = f(y)$. Verify that this is an equivalence relation.

(g) Let $\Sigma = \{a, b\}$ be an alphabet with two letters. Define $R \subset \Sigma^* \times \Sigma^*$ as follows. $(w, x) \in R$ iff $|wx|$ is even. R is an equivalence relation. The proof is similar to third example above. There are two equivalence classes: the strings of even length and the strings of odd length.

We can look at this relation differently. Define $R' \subset \Sigma^* \times \Sigma^*$ as follows. $(w, x) \in R$ iff for all $v \in \Sigma^*$, $|wv|$ and $|xv|$ have same parity that is they have either both even length or both odd length. Can you prove that $R = R'$? Look at R' carefully. For (w, x) to be its member we must test wv and xv for *all* strings. There are infinite number of strings. So how do we do that? We have to be a bit smart about it. *Hint*: v is either odd or even.

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