

6. Parameterized branching algorithms

COMP6741: Parameterized and Exact Computation

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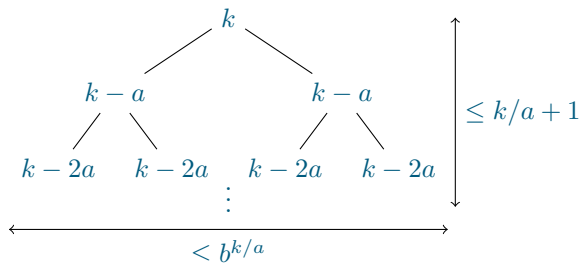
Semester 2, 2017

- 1 Running time analysis
- 2 Feedback Vertex Set
- 3 Maximum Leaf Spanning Tree
- 4 Further Reading

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Search trees

Recall: A **search tree** models the recursive calls of an algorithm.
For a b -way branching where the parameter k decreases by a at each recursive call, the number of nodes is at most $b^{k/a} \cdot (k/a + 1)$.



If k/a and b are upper bounded by a function of k , and the time spent at each node is **FPT** (typically, polynomial), then we get an **FPT** running time.

Recall: Measure Based Analysis

For more precise running time upper bounds:

Lemma 1 (Measure Analysis Lemma)

Let

- A be a branching algorithm
- $c \geq 0$ be a constant, and
- $\mu(\cdot), \eta(\cdot)$ be two measures for the instances of A ,

such that on input I , A calls itself recursively on instances I_1, \dots, I_k , but, besides the recursive calls, uses time $O((\eta(I))^c)$, such that

$$(\forall i) \quad \eta(I_i) \leq \eta(I) - 1, \text{ and} \quad (1)$$

$$2^{\mu(I_1)} + \dots + 2^{\mu(I_k)} \leq 2^{\mu(I)}. \quad (2)$$

Then A solves any instance I in time $O(\eta(I)^{c+1}) \cdot 2^{\mu(I)}$.

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Feedback Vertex Set

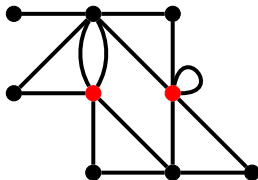
A **feedback vertex set** of a multigraph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G - S$ is acyclic.

FEEDBACK VERTEX SET

Input: Multigraph $G = (V, E)$, integer k

Parameter: k

Question: Does G have a feedback vertex set of size at most k ?



Simplification Rules

We apply the first **applicable**¹ simplification rule.

(Loop)

If G has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

¹A simplification rule is **applicable** if it modifies the instance.

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If E contains an edge uv more than twice, remove all but two copies of uv .

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(Degree-1)

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$.

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(Degree-1)

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$.

(Budget-exceeded)

If $k < 0$, then return **No**.

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Simplification Rules II

(Degree-2)

If $\exists v \in V$ with $d_G(v) = 2$, then denote $N_G(v) = \{u, w\}$ and set $G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\})$.

Simplification Rules II

(Degree-2)

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Lemma 2

(Degree-2) is sound.

Proof.

Suppose S is a feedback vertex set of G of size at most k . Let

$$S' = \begin{cases} S & \text{if } v \notin S \\ (S \setminus \{v\}) \cup \{u\} & \text{if } v \in S. \end{cases}$$

Now, $|S'| \leq k$ and S' is a feedback vertex set of G' since every cycle in G' corresponds to a cycle in G , with, possibly, the edge uw replaced by the path (u, v, w) .

Suppose S' is a feedback vertex set of G' of size at most k . Then, S' is also a feedback vertex set of G . □

Remaining issues

- A select–discard branching decreases k in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of k

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Idea:

- An acyclic graph has average degree < 2
- After applying simplification rules, G has average degree ≥ 3
- The selected feedback vertex set needs to be incident to many edges
- Does a feedback vertex set of size at most k contain at least one vertex among the $f(k)$ vertices of highest degree?

The fvs needs to be incident to many edges

Lemma 3

If S is a feedback vertex set of $G = (V, E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

Proof.

Since $F = G - S$ is acyclic, $|E(F)| \leq |V| - |S| - 1$.

Since every edge in $E \setminus E(F)$ is incident with a vertex of S , we have

$$\begin{aligned} |E| &= |E| - |E(F)| + |E(F)| \\ &\leq \left(\sum_{v \in S} d_G(v) \right) + (|V| - |S| - 1) \\ &= \left(\sum_{v \in S} (d_G(v) - 1) \right) + |V| - 1. \end{aligned}$$



The fvs needs to contain a high-degree vertex

Lemma 4

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G .

Every feedback vertex set of G of size at most k contains at least one vertex of H .

The fvs needs to contain a high-degree vertex

Lemma 4

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G .

Every feedback vertex set of G of size at most k contains at least one vertex of H .

Proof.

Suppose not. Let S be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

$$\begin{aligned} 2|E| - |V| &= \sum_{v \in V} (d_G(v) - 1) \\ &= \sum_{v \in H} (d_G(v) - 1) + \sum_{v \in V \setminus H} (d_G(v) - 1) \\ &\geq 3 \cdot \left(\sum_{v \in S} (d_G(v) - 1) \right) + \sum_{v \in S} (d_G(v) - 1) \\ &\geq 4 \cdot (|E| - |V| + 1) \\ \Leftrightarrow 3|V| &\geq 2|E| + 4. \end{aligned}$$

But this contradicts the fact that every vertex of G has degree at least 3. □

Theorem 5

FEEDBACK VERTEX SET *can be solved in $O^*((3k)^k)$ time.*

Proof (sketch).

- Exhaustively apply the simplification rules.
- The branching rule computes H of size $3k$, and branches into subproblems $(G - v, k - 1)$ for each $v \in H$.



Current best: $O^*(3.619^k)$ [Kociumaka, Pilipczuk, 2014]

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Maximum Leaf Spanning Tree

A **leaf** of a tree is a vertex with degree 1. A **spanning tree** in a graph $G = (V, E)$ is a subgraph of G that is a tree and has $|V|$ vertices.

MAXIMUM LEAF SPANNING TREE

Input: connected graph G , integer k

Parameter: k

Question: Does G have a spanning tree with at least k leaves?

Property

A k -leaf tree in G is a subgraph of G that is a tree with at least k leaves.

A k -leaf spanning tree in G is a spanning tree in G with at least k leaves.

Lemma 6

Let $G = (V, E)$ be a connected graph.

G has a k -leaf tree $\Leftrightarrow G$ has a k -leaf spanning tree.

Proof.

(\Leftarrow): trivial

(\Rightarrow): Let T be a k -leaf tree in G . By induction on $x := |V| - |V(T)|$, we will show that T can be extended to a k -leaf spanning tree in G .

Base case: $x = 0$ ✓.

Induction: $x > 0$, and assume the claim is true for all $x' < x$. Choose $uv \in E$ such that $u \in V(T)$ and $v \notin V(T)$. Since $T' := (V(T) \cup \{v\}, E(T) \cup \{uv\})$ has $\geq k$ leaves and $< x$ external vertices, it can be extended to a k -leaf spanning tree in G by the induction hypothesis. \square

Strategy

- The branching algorithm will check whether G has a k -leaf tree.
- A tree with ≥ 3 vertices has at least one **internal** (= non-leaf) vertex.
- “Guess” an internal vertex r , i.e., do a $|V|$ -way branching fixing an initial internal vertex r .

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- In any branch, the algorithm has computed
 - T – a tree in G
 - I – the internal vertices of T , with $r \in I$
 - B – a subset of the leaves of T where T may be extended: the boundary set
 - L – the remaining leaves of T
 - X – the external vertices $V \setminus V(T)$

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- A tree with ≥ 3 vertices has at least one **internal** (= non-leaf) vertex.
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- In any branch, the algorithm has computed
 - T – a tree in G
 - I – the internal vertices of T , with $r \in I$
 - B – a subset of the leaves of T where T may be extended: the boundary set
 - L – the remaining leaves of T
 - X – the external vertices $V \setminus V(T)$
- The question is whether T can be extended to a k -leaf tree where all the vertices in L are leaves.

Simplification Rules

Apply the first applicable simplification rule:

(Halt-Yes)

If $|L| + |B| \geq k$, then return **YES**.

(Halt-No)

If $|B| = 0$, then return **No**.

(Non-extendable)

If $\exists v \in B$ with $N_G(v) \cap X = \emptyset$, then move v to L .

Lemma 7 (Branching Lemma)

Suppose $u \in B$ and there exists a k -leaf tree T' extending T where u is an internal vertex.

Then, there exists a k -leaf tree T'' extending $(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$.

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Proof.

Start from $T'' \leftarrow T'$ and perform the following operation for each $v \in N_G(u) \cap X$. If $v \notin V(T')$, then add the vertex v and the edge uv .

Otherwise, add the edge uv , creating a cycle C in T and remove the other edge of C incident to v . This does not decrease the number of leaves, since it only increases the number of edges incident to u , and u was already internal. \square

Lemma 8 (Follow Path Lemma)

Suppose $u \in B$ and $|N_G(u) \cap X| = 1$. Let $N_G(u) \cap X = \{v\}$.

If there exists a k -leaf tree extending T where u is internal, but no k -leaf tree extending T where u is a leaf, then there exists a k -leaf tree extending T where both u and v are internal.

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Proof.

Suppose not, and let T' be a k -leaf tree extending T where u is internal and v is a leaf. But then, $T' - v$ is a k -leaf tree as well. \square

- Apply simplification rules
- Select $u \in B$. Branch into
 - $u \in L$
 - $u \in I$. In this case, add $X \cap N_G(u)$ to B (Branching Lemma). In the special case where $|X \cap N_G(u)| = 1$, denote $\{v\} = X \cap N_G(u)$, make v internal, and add $N_G(v) \cap X$ to B , continuing the same way until reaching a vertex with at least 2 neighbors in X (Follow Path Lemma).

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 - $u \in L$
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- In one branch, a vertex moves from B to L ; in the other branch, $|B|$ increases by at least 1.

Running time analysis

- Measure $\mu := 2k - 2|L| - |B| \geq 0$.
- Branch where $u \in L$:
 - $|B|$ decreases by 1, $|L|$ increases by 1
 - μ decreases by 1
- Branch where $u \in I$.
 - u moves from B to I
 - ≥ 2 vertices move from X to B
 - μ decreases by at least 1

- Binary search tree
- Height $\leq \mu \leq 2k$

Result for Maximum Leaf Spanning Tree

Theorem 9 ([Kneis, Langer, Rossmanith, 2011])

MAXIMUM LEAF SPANNING TREE *can be solved in $O^*(4^k)$ time.*

Current best: $O^*(3.72^k)$ [Daligault, Gutin, Kim, Yeo, 2010]

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- Chapter 3, *Bounded Search Trees* in Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, MichałPilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer, 2015.
- Chapter 3, *Bounded Search Trees* in Rodney G. Downey and Michael R. Fellows. *Fundamentals of Parameterized Complexity*. Springer, 2013.
- Chapter 8, *Depth-Bounded Search Trees* in Rolf Niedermeier. *Invitation to Fixed Parameter Algorithms*. Oxford University Press, 2006.