2a. Kernelization

COMP6741: Parameterized and Exact Computation

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Outline

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Kernelization

Definition 1

A kernelization (kernel) for a parameterized problem Π is a **polynomial time** algorithm, which, for any instance I of Π with parameter k, produces an **equivalent** instance I' of Π with parameter k' such that $|I'| \leq f(k)$ and $k' \leq f(k)$ for a computable function f.
We refer to the function f as the size of the kernel.

Fixed-parameter tractability

Definition 2

A parameterized problem Π is fixed-parameter tractable (FPT) if there is an algorithm solving Π in time $f(k) \cdot \text{poly}(n)$, where n is the instance size, k is the parameter, poly is a polynomial function, and f is a computable function.

Theorem 3

Let Π be a decidable parameterized problem.
Π has a kernelization $\Leftrightarrow$ Π is FPT.

Outline

A Hamiltonian cycle of G is a subgraph of G that is a cycle on $|V(G)|$ vertices.

vc-Hamiltonian Cycle

Input:	A graph $G = (V, E)$.
Parameter:	$k = vc(G)$, the size of a smallest vertex cover of G.
Question:	Does G have a Hamiltonian cycle?

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

Issue: We do not actually know a vertex cover of size k.

- Obtain a vertex cover of size $\leq 2k$ by applying Vertex Cover-kernelizations to $(G,0),(G,1),\ldots$ until the first instance where no trivial No-instance is returned.
- If C is a vertex cover of size $\leq 2k$, then $I = V \setminus C$ is an independent set of size $\geq |V| - 2k$.
- No two consecutive vertices in the Hamiltonian Cycle can be in I.
- A kernel with $\leq 4k$ vertices can now be obtained with the following simplification rule.

(Too-large)

Compute a vertex cover C of size $\leq 2k$ in polynomial time.
If $2|C| < |V|$, then return No

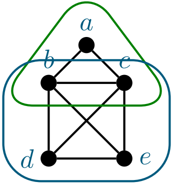
Outline

Edge Clique Cover

Definition 4

An edge clique cover of a graph $G = (V, E)$ is a set of cliques in G covering all its edges.
In other words, if $\mathcal{C} \subseteq 2^V$ is an edge clique cover then each $S \in \mathcal{C}$ is a clique in G and for each $\{u, v\} \in E$ there exists an $S \in \mathcal{C}$ such that $u, v \in S$.

Example: $\{\{a, b, c\}, \{b, c, d, e\}\}$ is an edge clique cover for this graph.



Edge Clique Cover

Edge Clique Cover

Input:	A graph $G = (V, E)$ and an integer k
Parameter:	k
Question:	Does G have an edge clique cover of size at most k?

The size of an edge clique cover $\mathcal{C}$ is the number of cliques contained in $\mathcal{C}$ and is denoted $|\mathcal{C}|$.

Helpful properties

Definition 4

A clique S in a graph G is a maximal clique if there is no other clique S' in G with $S \subset S'$.

Lemma 5

A graph G has an edge clique cover $\mathcal{C}$ of size at most k if and only if G has an edge clique cover $\mathcal{C}'$ of size at most k such that each $S \in \mathcal{C}'$ is a maximal clique.

Proof sketch.

$(\Rightarrow)$: Replace each clique $S \in \mathcal{C}$ by a maximal clique S' with $S \subseteq S'$.
$(\Leftarrow)$: Trivial, since $\mathcal{C}'$ is an edge clique cover of size at most k. □

Simplification rules for EDGE CLIQUE COVER

Thought experiment: Imagine a very large instance where the parameter is tiny. How can you simplify such an instance?

Simplification rules for Edge Clique Cover II

The instance could have many degree-0 vertices.

(Isolated)

If there exists a vertex $v \in V$ with $d_G(v) = 0$, then set $G \leftarrow G - v$.

Lemma 6

(Isolated) is sound.

Proof sketch.

Since no edge is incident to v, a smallest edge clique cover for $G - v$ is a smallest edge clique cover for G, and vice-versa. □

Simplification rules for Edge Clique Cover II

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(Isolated-Edge)

If $\exists uv \in E$ such that $d_G(u) = d_G(v) = 1$, then set $G \leftarrow G - \{u, v\}$ and $k \leftarrow k - 1$.

Simplification rules for Edge Clique Cover III

(Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

Lemma 7

(Twins) is sound.

Simplification rules for EDGE CLIQUE COVER III

(Twins)

If $\exists u, v \in V$, $u \neq v$, such that $N_G[u] = N_G[v]$, then set $G \leftarrow G - v$.

Lemma 7

(Twins) is sound.

Proof.

We need to show that G has an edge clique cover of size at most k if and only if $G - v$ has an edge clique cover of size at most k.
$(\Rightarrow)$: If $\mathcal{C}$ is an edge clique cover of G of size at most k, then $\{S \setminus \{v\} : S \in \mathcal{C}\}$ is an edge clique cover of $G - v$ of size at most k.
$(\Leftarrow)$: Let $\mathcal{C}'$ be an edge clique cover of $G - v$ of size at most k. Partition $\mathcal{C}$ into $\mathcal{C}_u = \{S \in \mathcal{C} : u \in S\}$ and $\mathcal{C}_{\neg u} = \mathcal{C} \setminus \mathcal{C}_u$. Note that each set in $\mathcal{C}'_u = \{S \cup \{v\} : S \in \mathcal{C}_u\}$ is a clique since $N_G[u] = N_G[v]$ and that each edge incident to v is contained in at least one of these cliques. Now, $\mathcal{C}'_u \cup \mathcal{C}_{\neg u}$ is an edge clique cover of G of size at most k. □

Simplification rules for Edge Clique Cover IV

(Size-V)

If the previous simplification rules do not apply and $|V| > 2^k$, then return No.

Lemma 8

(Size-V) is sound.

Simplification rules for Edge Clique Cover IV

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Proof.

For the sake of contradiction, assume neither (Isolated) nor (Twins) are applicable, $|V| > 2^k$, and G has an edge clique cover $\mathcal{C}$ of size at most k. Since $2^{\mathcal{C}}$ (the set of all subsets of $\mathcal{C}$) has size at most 2^k, and every vertex belongs to at least one clique in $\mathcal{C}$ by (Isolated), we have that there exists two vertices $u, v \in V$ such that $\{S \in \mathcal{C} : u \in S\} = \{S \in \mathcal{C} : v \in S\}$. But then, $N_G[u] = \bigcup_{S \in \mathcal{C} : u \in S} S = \bigcup_{S \in \mathcal{C} : v \in S} S = N_G[v]$, contradicting that (Twin) is not applicable. □

Kernel for Edge Clique Cover

Theorem 9

Edge Clique Cover *has a kernel with* $O(2^k)$ *vertices and* $O(4^k)$ *edges.*

Corollary 10

Edge Clique Cover *is* FPT.

Outline

Frequent issues designing simplification rules

Issue 1: A kernelization needs to produce an instance of the same problem.

How could we turn the following lemma into a simplification rule?

Lemma 11

If there is an edge $\{u, v\} \in E$ *such that* $S = N_G[u] \cap N_G[v]$ *is a clique, then there is a smallest edge clique cover* $\mathcal{C}$ *with* $S \in \mathcal{C}$.

Proof.

By Lemma 5, we may assume the clique covering the edge $\{u, v\}$ is a maximal clique. But, S is the unique maximal clique covering $\{u, v\}$. □

Frequent issues designing simplification rules

(Neighborhood-Clique)

If there exists $\{u, v\} \in E$ such that $S = N_G[u] \cap N_G[v]$ is a clique, then ...???

Edges with both endpoints in $S \setminus \{u, v\}$ are covered by S but might still be needed in other cliques.

Frequent issues designing simplification rules

We could design a kernelization for a more general problem.

GENERALIZED EDGE CLIQUE COVER

Input:	A graph $G = (V, E)$, a set of edges $R \subseteq E$, and an integer k
Parameter:	k
Question:	Is there a set $\mathcal{C}$ of at most k cliques in G such that each $e \in R$ is contained in at least one of these cliques?

(Neighborhood-Clique)

If there exists $\{u, v\} \in R$ such that $S = N_G[u] \cap N_G[v]$ is a clique, then set $G \leftarrow (V, E \setminus \{u, v\})$, $R \leftarrow R \setminus \{\{x, y\} : x, y \in S\}$, and $k \leftarrow k - 1$.

Frequent issues designing simplification rules

Issue 2: A proposed simplification rule might not be sound.

Consider the following simplification rule for VERTEX COVER.

(Degk)

If $\exists v \in V$ such that $d_G(v) \geq k$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

To show that a simplification rule is not sound, we exhibit a counter-example.

Frequent issues designing simplification rules

Issue 3: A problem might be FPT, but only an exponential kernel might be known / possible to achieve.