COMP2111 Week 7
Term 1, 2019
State machines

Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata

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Motivation: Models of computation

State machines model step-by-step processes:

- Set of "states", possibly including a designated "start state"
- For each state, a set of actions detailing how to move (transition) to other states

Example

The semantics of a program in $\mathcal{L}$:

- States: functions from variables to numerical values
- Transitions: defined by the program

Motivation: Models of computation

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Example

A chess solving engine

- States: Board positions
- Transitions: Legal moves

Motivation: Models of computation

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Example

"Stateful" communication protocols: e.g. SMTP

- States: Stages of communication
- Transitions: Determined by commands given (e.g. HELO, DATA, etc)

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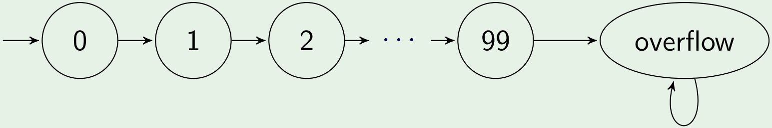
Example

A bounded counter that counts from 0 to 99 and overflows at 100:



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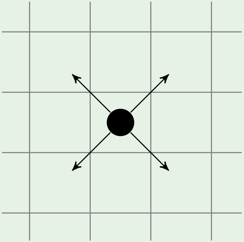
Example

A robot that moves diagonally



States: Locations

Transitions: Moves

Motivation: Models of computation

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Example

Die Hard jug problem: Given jugs of 3L and 5L, measure out exactly 4L.

- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)

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Definitions

A **transition system** is a pair $(S, \rightarrow)$ where:

- S is a set (of **states**), and
- $\rightarrow \subseteq S \times S$ is a (**transition**) **relation**.

If $(s, s') \in \rightarrow$ we write $s \rightarrow s'$.

- S may have a designated **start state**, $s_0 \in S$
- S may have designated **final states**, $F \subseteq S$
- The transitions may be **labelled** by elements of a set Λ:
  - $\rightarrow \subseteq S \times \Lambda \times S$
  - $(s, a, s') \in \rightarrow$ is written as $s \xrightarrow{a} s'$
- If $\rightarrow$ is a function we say the system is **deterministic**, otherwise it is **non-deterministic**

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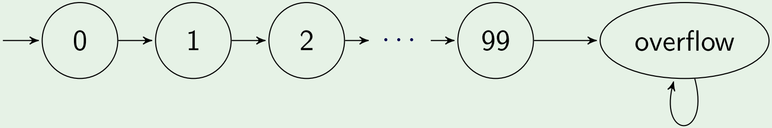
Example: Bounded counter

Example

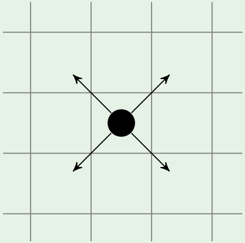
A bounded counter that counts from 0 to 99 and overflows at 100:



- $S = \{0, 1, \ldots, 99, \text{overflow}\}$
- $\rightarrow = \begin{array}{l} \{(i, i+1) \ : \ 0 \leq i < 99\} \\ \cup \ \{(99, \text{overflow})\} \\ \cup \ \{(\text{overflow}, \text{overflow})\} \end{array}$
- $s_0 = 0$
- Deterministic

Example: Diagonally moving robot

Example



States: Locations

Transitions: Moves

Example: Diagonally moving robot

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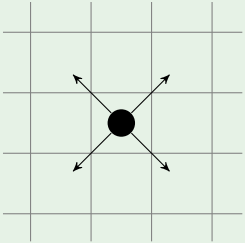


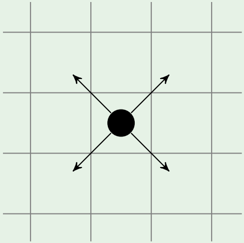
$S = \mathbb{Z} \times \mathbb{Z}$

$(x, y) \to (x \pm 1, y \pm 1)$

Non-deterministic

Example: Diagonally moving robot

Example



$S = \mathbb{Z} \times \mathbb{Z}$

$\Lambda = \{NW, NE, SW, SE\}$

$(x, y) \xrightarrow{NW} (x-1, y+1)$

$(x, y) \xrightarrow{NE} (x+1, y+1)$

$(x, y) \xrightarrow{SW} (x-1, y-1)$

$(x, y) \xrightarrow{SE} (x+1, y-1)$

Deterministic

Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- States: Defined by amount of water in each jug
- Start state: No water in both jugs
- Transitions: Pouring water (in, out, jug-to-jug)

Example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

- $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} \, : \, 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0, 0)$
- $\rightarrow$ given by
  - $(i, j) \rightarrow (0, j)$ [empty 5L jug]
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  - $(i, j) \rightarrow (5, j)$ [fill 5L jug]
  - $(i, j) \rightarrow (i, 3)$ [fill 3L jug]
  - $(i, j) \rightarrow (i + j, 0)$ if $i + j \leq 5$ [empty 3L jug into 5L jug]
  - $(i, j) \rightarrow (0, i + j)$ if $i + j \leq 3$ [empty 5L jug into 3L jug]
  - $(i, j) \rightarrow (5, j - 5 + i))$ if $i + j \geq 5$ [fill 5L jug from 3L jug]
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Runs and reachability

Given a transition system $(S, \to)$ and states $s, s' \in S$,

- a **run** from s is a (possibly infinite) sequence $s_1, s_2, \ldots$ such that $s = s_1$ and $s_i \to s_{i+1}$ for all $i \geq 1$.
- we say s' is **reachable** from s, written $s \to^* s'$, if (s, s') is in the transitive closure of $\to$.

NB

s' is reachable from s if there is a run from s which contains s'.

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Safety and Liveness

Common problem (Safety)

Will a transition system always avoid a particular state or states?
Equivalently, can a transition system reach a particular state or states?

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Reachability example: Die Hard jug problem

Example

Given jugs of 3L and 5L, measure out exactly 4L.

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- Transition relation: $(i, j) \to (0, j)$ etc.

Is $(4, 0)$ reachable from $(0, 0)$?

Yes:

$$\begin{array}{ccccc} (0,0) & \to & (0,3) & \to & (3,0) \\ & & & & \downarrow \\ (0,1) & \leftarrow & (5,1) & \leftarrow & (3,3) \\ \downarrow & & & & \\ (1,0) & \to & (1,3) & \to & (4,0) \end{array}$$

Reachability example: Die Hard jug problem

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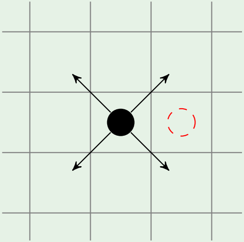
Safety example: Diagonally moving robot

Example



Starting at $(0, 0)$

Can the robot get to $(0, 1)$?

Safety example: Diagonally moving robot

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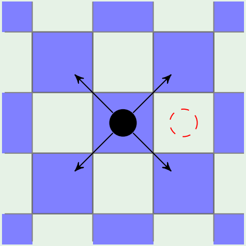

Starting at $(0, 0)$

Can the robot get to $(0, 1)$?

Safety example: Diagonally moving robot

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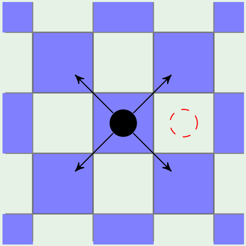

Starting at $(0, 0)$

Can the robot get to $(0, 1)$? No

Safety example: Diagonally moving robot

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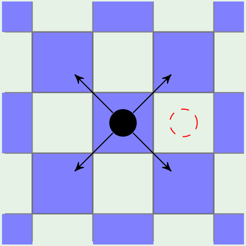

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$\text{isBlue}((m,n)) := 2|(m+n)$

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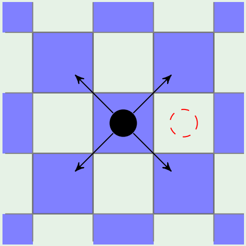

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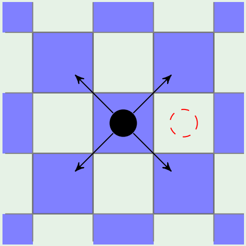

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$\text{isBlue}((0, 0))$ and $\neg\text{isBlue}((0, 1))$

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The invariant principle

A **preserved invariant** of a transition system is a unary predicate φ on states such that if $\varphi(s)$ holds and $s \to s'$ then $\varphi(s')$ holds.

Invariant principle

If a preserved invariant holds at a state s, then it holds for all states reachable from s.

Proof:

The invariant principle

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Invariant principle

If a preserved invariant holds at a state s, then it holds for all states reachable from s.

Proof:

Invariant example: Modified Die Hard problem

Example

Given jugs of 3L and 6L, measure out exactly 4L.

- States: $S = \{(i, j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq 6 \text{ and } 0 \leq j \leq 3\}$
- Transition relation: $(i, j) \rightarrow (0, j)$ etc.

Is $(4, 0)$ reachable from $(0, 0)$?

No. Consider $\varphi((i, j)) = (3|i) \wedge (3|j)$.

Invariant example: Modified Die Hard problem

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Partial correctness

Let $(S, \rightarrow, s_0, F)$ be a transition system with start state s_0 and final states F and a φ be a unary predicate on S. We say the system is **partially correct for** φ if $\varphi(s')$ holds for all states $s' \in F$ that are reachable from s_0.

NB

Partial correctness does not guarantee a transition system will reach a final state.

Partial correctness example: Fast exponentiation

Example

Consider the following program in $\mathcal{L}$:

$$
\begin{array}{l}
x := m; \\
y := n; \\
r := 1; \\
\textbf{while } y > 0 \textbf{ do} \\
\quad \textbf{if } 2 | y \textbf{ then} \\
\quad\quad y := y/2 \\
\quad \textbf{else} \\
\quad\quad y := (y-1)/2; \\
\quad\quad r := r * x \\
\quad \textbf{fi}; \\
\quad x := x * x \\
\textbf{od}
\end{array}
$$

Partial correctness example: Fast exponentiation

Example

- States: Functions from $\{m, n, x, y, r\}$ to $\mathbb{N}$
- Transitions: Effect of each line of code:
  - $(x, y, r) \rightarrow (x^2, y/2, r)$ if y is even
  - $(x, y, r) \rightarrow (x^2, (y-1)/2, rx)$ if y is odd
- Start state: $(m, n, 1)$
- Final states: $\{(x, 0, r) : x, r \in \mathbb{N}\}$

Goal: Show partial correctness for $\varphi((x, y, r)) := (r = m^n)$

Show $\psi((x, y, r)) := (rx^y = m^n)$ is a preserved invariant...

How can we show total correctness?

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each line of code:
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Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
- Transitions: Effect of each iteration of while loop:
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- Final states: $\{(x, 0, r) \ : \ x, r \in \mathbb{N}\}$

Goal: Show partial correctness for $\varphi((x, y, r)) := (r = m^n)$

Show $\psi((x, y, r)) := (rx^y = m^n)$ is a preserved invariant...

How can we show total correctness?

Partial correctness example: Fast exponentiation

Example

- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
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How can we show total correctness?

Total correctness

A transition system $(S, \to)$ **terminates** from a state $s \in S$ if there is an $N \in \mathbb{N}$ such that all runs from s have length at most N.

A transition system is **totally correct for a unary predicate** φ, if it terminates (from s_0) and φ holds in the last state of every run.

Derived variables

In a transition system $(S, \to)$, a **derived variable** is a function $f : S \to \mathbb{R}$.

A derived variable is **strictly decreasing** if $s \to s'$ implies $f(s') < f(s)$.

Theorem

If f is an $\mathbb{N}$-valued, strictly decreasing derived variable, then the length of any run from s is at most $f(s)$.

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Termination example: Fast exponentiation

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- States: $(x, y, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$
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Derived variable: $f((x, y, r)) = y$

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Summary

- Motivation
- Definitions
- The invariant principle
- Partial correctness and termination
- Input and output
- Finite automata

Interaction with the environment

We can model the system interacting with an external entity via inputs (Σ) and outputs (Γ) by using **labelled transitions**:
$\rightarrow \subseteq S \times \Lambda \times S$ where $\Lambda = \Sigma \times \Gamma$

Two main categories of input/output transition systems:

Acceptors: Accept/reject a sequence of inputs (Relations)

Transducers: Take a sequence of inputs and produce a sequence of outputs (Functions)

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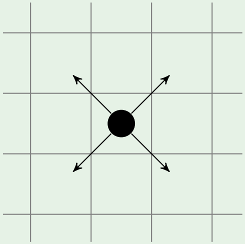
Acceptor example: Diagonally moving robot

Example



$S = \mathbb{Z} \times \mathbb{Z}$

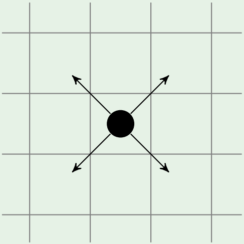
$s_0 = (0, 0)$

$(x, y) \xrightarrow{NW} (x - 1, y + 1)$
$(x, y) \xrightarrow{NE} (x + 1, y + 1)$
$(x, y) \xrightarrow{SW} (x - 1, y - 1)$
$(x, y) \xrightarrow{SE} (x + 1, y - 1)$

Accept if $(2, 2)$ reached

Acceptor example: Diagonally moving robot

Example



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Accepted sequences:
NE, NE

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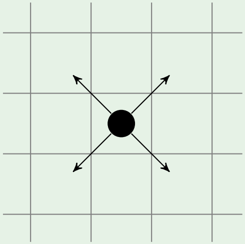

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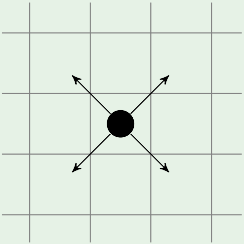
Accept if $(2, 2)$ reached

Accepted sequences:
NE, NE
NE, SE, NE, NW

Acceptor example: Diagonally moving robot

Example



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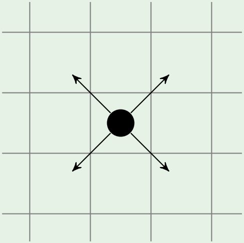
Accept if $(2, 2)$ reached

Accepted sequences:
NE, NE
NE, SE, NE, NW
NE, NE, NE, SW ...

Transducer example: Diagonally moving robot

Example



$S = \mathbb{Z} \times \mathbb{Z}$

$s_0 = (0, 0)$

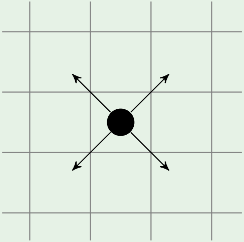
$$(x, y) \xrightarrow{NW/x} (x-1, y+1)$$
$$(x, y) \xrightarrow{NE/x} (x+1, y+1)$$
$$(x, y) \xrightarrow{SW/x} (x-1, y-1)$$
$$(x, y) \xrightarrow{SE/x} (x+1, y-1)$$

Input direction

Output x-coordinate

Transducer example: Diagonally moving robot

Example



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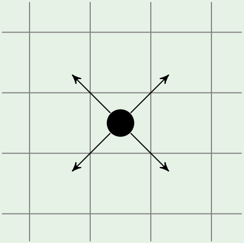
Input direction
Output x-coordinate

Input: NE, SE, NE, NW
Output: $1, 2, 3, 2$

Transducer example: Diagonally moving robot

Example



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$$(x, y) \xrightarrow{SE/y} (x + 1, y - 1)$$

Input direction
Output y-coordinate

Input: NE, SE, NE, NW
Output: $1, 0, 1, 2$

Acceptor example: Die Hard jug problem

Example

- $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq 5 \text{ and } 0 \leq j \leq 3\}$
- $s_0 = (0,0)$
- $\rightarrow$ given by
  - $(i,j) \xrightarrow{\text{E5}} (0,j)$ [empty 5L jug]
  - $(i,j) \xrightarrow{\text{E3}} (i,0)$ [empty 3L jug]
  - $(i,j) \xrightarrow{\text{F5}} (5,j)$ [fill 5L jug]
  - $(i,j) \xrightarrow{\text{F3}} (i,3)$ [fill 3L jug]
  - $(i,j) \xrightarrow{\text{E35}} (i+j,0)$ if $i+j \leq 5$ [empty 3L jug into 5L jug]
  - $(i,j) \xrightarrow{\text{E53}} (0,i+j)$ if $i+j \leq 3$ [empty 5L jug into 3L jug]
  - $(i,j) \xrightarrow{\text{F53}} (5,j-5+i))$ if $i+j \geq 5$ [fill 5L jug from 3L jug]
  - $(i,j) \xrightarrow{\text{F35}} (i-3+j,3)$ if $i+j \geq 3$ [fill 3L jug from 5L jug]
- Accept if $(4,0)$ is reached: e.g. F3, E35, F3, F53, E5, E35, F3, E35

Acceptor example: Die Hard jug problem

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- $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \ : \ 0 \leq i \leq 5$ and $0 \leq j \leq 3\}$
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- Accept if $(4,0)$ is reached: e.g. F3, E35, F3, F53, E5, E35, F3, E35

ϵ-transitions

It can be useful to allow the system to transition without taking input or producing output. We use the special symbol ϵ to denote such transitions.

Formal definitions

An **acceptor** is a $\Sigma \cup \{\epsilon\}$-labelled transition system $A = (S, \rightarrow, \Sigma, s_0, F)$ with a start state $s_0 \in S$ and a set of final states $F \subseteq S$.

A **transducer** is a $(\Sigma \cup \{\epsilon\}) \times (\Gamma \cup \{\epsilon\})$-labelled transition system $T = (S, \rightarrow, \Sigma, s_0, F)$ with a start state $s_0 \in S$ and a set of final states $F \subseteq S$.

Summary

- Motivation
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Finite state transition systems

State transition systems with a finite set of states are particularly useful in Computer Science.

Acceptors: **Finite state automata**

Transducers: **Mealy machines**