5a. Branching algorithms
COMP6741: Parameterized and Exact Computation

Serge Gaspers

School of Computer Science and Engineering, UNSW Sydney, Australia

19T3

Outline

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Branching Algorithm

Branching Algorithm

- Selection: Select a local configuration of the problem instance
- Recursion: Recursively solve subinstances
- Combination: Compute a solution of the instance based on the solutions of the subinstances

- Halting rule: 0 recursive calls
- Simplification rule: 1 recursive call
- Branching rule: ≥ 2 recursive calls

Example: Our first Vertex Cover algorithm

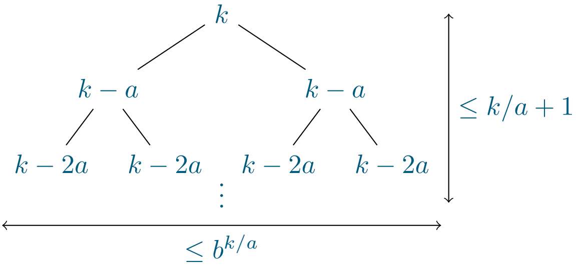
Algorithm vc1(G, k);

if $E = \emptyset$ **then** `// all edges are covered`
  return Yes
else if $k \leq 0$ **then** `// we cannot select any vertex`
  return No
else
  Select an edge $uv \in E$;
  return vc1$(G - u, k - 1)$ $\vee$ vc1$(G - v, k - 1)$

Outline

Search trees

Recall: A search tree models the recursive calls of an algorithm.
For a b-way branching where the parameter k decreases by a at each recursive call, the number of nodes is at most $b^{k/a} \cdot (k/a+1)$.



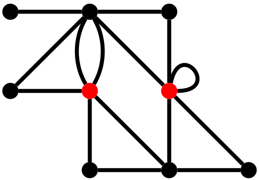
If k/a and b are upper bounded by a function of k, and the time spent at each node is FPT (typically, polynomial), then we get an FPT running time.

Outline

Feedback Vertex Set

A feedback vertex set of a multigraph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that $G - S$ is acyclic.

FEEDBACK VERTEX SET
Input: Multigraph $G = (V, E)$, integer k
Parameter: k
Question: Does G have a feedback vertex set of size at most k?



Simplification Rules

We apply the first applicable[1] simplification rule.

(Finished)

If G is acyclic and $k \geq 0$, then return YES.

(Budget-exceeded)

If $k < 0$, then return NO.

[1] A simplification rule is applicable if it modifies the instance.

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(Loop)

If G has a loop $vv \in E$, then set $G \leftarrow G - v$ and $k \leftarrow k - 1$.

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(Multiedge)

If E contains an edge uv more than twice, remove all but two copies of uv.

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Simplification Rules II

(Degree-1)

If $\exists v \in V$ with $d_G(v) \leq 1$, then set $G \leftarrow G - v$.

Simplification Rules III

(Degree-2)

If $\exists v \in V$ with $d_G(v) = 2$, then denote $N_G(v) = \{u, w\}$ and set $G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\})$.

Simplification Rules III

(Degree-2)

If $\exists v \in V$ with $d_G(v) = 2$, then denote $N_G(v) = \{u, w\}$ and set $G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\})$.

Lemma 1

(Degree-2) is sound.

Proof.

Suppose S is a feedback vertex set of G of size at most k. Let

$$S' = \begin{cases} S & \text{if } v \notin S \\ (S \setminus \{v\}) \cup \{u\} & \text{if } v \in S. \end{cases}$$

Now, $|S'| \le k$ and S' is a feedback vertex set of G' since every cycle in G' corresponds to a cycle in G, with, possibly, the edge uw replaced by the path (u, v, w).

Suppose S' is a feedback vertex set of G' of size at most k. Then, S' is also a feedback vertex set of G. $\square$

Remaining issues

- A select–discard branching decreases k in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of k

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Idea:

- An acyclic graph has average degree < 2
- After applying simplification rules, G has average degree ≥ 3
- The selected feeback vertex set needs to be incident to many edges
- Does a feedback vertex set of size at most k contain at least one vertex among the $f(k)$ vertices of highest degree?

The fvs needs to be incident to many edges

Lemma 2

If S is a feedback vertex set of $G = (V, E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

The fvs needs to be incident to many edges

Lemma 2

If S is a feedback vertex set of $G = (V, E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

Proof.

Since $F = G - S$ is acyclic, $|E(F)| \leq |V| - |S| - 1$.
Since every edge in $E \setminus E(F)$ is incident with a vertex of S, we have

$$\begin{aligned} |E| &= |E| - |E(F)| + |E(F)| \\ &\leq \left(\sum_{v \in S} d_G(v) \right) + (|V| - |S| - 1) \\ &= \left(\sum_{v \in S} (d_G(v) - 1) \right) + |V| - 1. \end{aligned}$$

□

The fvs needs to contain a high-degree vertex

Lemma 3

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G.
Every feedback vertex set of G of size at most k contains at least one vertex of H.

The fvs needs to contain a high-degree vertex

Lemma 3

Let G be a graph with minimum degree at least 3 and let H denote a set of $3k$ vertices of highest degree in G.
Every feedback vertex set of G of size at most k contains at least one vertex of H.

Proof.

Suppose not. Let S be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

$$\begin{aligned}
2|E| - |V| &= \sum_{v \in V} (d_G(v) - 1) \\
&= \sum_{v \in H} (d_G(v) - 1) + \sum_{v \in V \setminus H} (d_G(v) - 1) \\
&\geq 3 \cdot (\sum_{v \in S} (d_G(v) - 1)) + \sum_{v \in S} (d_G(v) - 1) \\
&\geq 4 \cdot (|E| - |V| + 1) \\
\Leftrightarrow \quad 3|V| &\geq 2|E| + 4.
\end{aligned}$$

But this contradicts the fact that every vertex of G has degree at least 3. $\square$

Algorithm for Feedback Vertex Set

Theorem 4

FEEDBACK VERTEX SET *can be solved in* $O^*((3k)^k)$ *time.*

Proof (sketch).

- Exhaustively apply the simplification rules.
- The branching rule computes H of size $3k$, and branches into subproblems $(G - v, k - 1)$ for each $v \in H$.

□

Current best:
$O^*(3.460^k)$ deterministic [IK19],
$O^*(2.7^k)$ time randomized [LN19]

Outline

Maximum Leaf Spanning Tree

A leaf of a tree is a vertex with degree 1. A spanning tree in a graph $G = (V, E)$ is a subgraph of G that is a tree and has $|V|$ vertices.

MAXIMUM LEAF SPANNING TREE	
Input:	connected graph G, integer k
Parameter:	k
Question:	Does G have a spanning tree with at least k leaves?

Property

A k-leaf tree in G is a subgraph of G that is a tree with at least k leaves.
A k-leaf spanning tree in G is a spanning tree in G with at least k leaves.

Lemma 5

Let $G = (V, E)$ be a connected graph.
G has a k-leaf tree $\Leftrightarrow$ G has a k-leaf spanning tree.

Proof.

$(\Leftarrow)$: trivial
$(\Rightarrow)$: Let T be a k-leaf tree in G. By induction on $x := |V| - |V(T)|$, we will show that T can be extended to a k-leaf spanning tree in G.
Base case: $x = 0$ $\checkmark$.
Induction: $x > 0$, and assume the claim is true for all $x' < x$. Choose $uv \in E$ such that $u \in V(T)$ and $v \notin V(T)$. Since $T' := (V(T) \cup \{v\}, E(T) \cup \{uv\})$ has $\geq k$ leaves and $< x$ external vertices, it can be extended to a k-leaf spanning tree in G by the induction hypothesis. $\square$

Strategy

- The branching algorithm will check whether G has a k-leaf tree.
- A tree with ≥ 3 vertices has at least one internal (= non-leaf) vertex.
- "Guess" an internal vertex r, i.e., do a $|V|$-way branching fixing an initial internal vertex r.

Strategy

- The branching algorithm will check whether G has a k-leaf tree.
- A tree with ≥ 3 vertices has at least one internal (= non-leaf) vertex.
- "Guess" an internal vertex r, i.e., do a $|V|$-way branching fixing an initial internal vertex r.
- In any branch, the algorithm has computed
  - T – a tree in G
  - I – the internal vertices of T, with $r \in I$
  - B – a subset of the leaves of T where T may be extended: the boundary set
  - L – the remaining leaves of T
  - X – the external vertices $V \setminus V(T)$

Strategy

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  - T – a tree in G
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  - B – a subset of the leaves of T where T may be extended: the boundary set
  - L – the remaining leaves of T
  - X – the external vertices $V \setminus V(T)$
- The question is whether T can be extended to a k-leaf tree where all the vertices in L are leaves.

Simplification Rules

Apply the first applicable simplification rule:

(Halt-Yes)

If $|L| + |B| \geq k$, then return $\textsc{Yes}$.

(Halt-No)

If $|B| = 0$, then return $\textsc{No}$.

(Non-extendable)

If $\exists v \in B$ with $N_G(v) \cap X = \emptyset$, then move v to L.

Branching Lemma

Lemma 6 (Branching Lemma)

Suppose $u \in B$ and there exists a k-leaf tree T' extending T where u is an internal vertex.
Then, there exists a k-leaf tree T'' extending
$(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$.

Branching Lemma

Lemma 6 (Branching Lemma)

Suppose $u \in B$ *and there exists a* k*-leaf tree* T' *extending* T *where* u *is an internal vertex.*
Then, there exists a k*-leaf tree* T'' *extending* $(V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})$.

Proof.

Start from $T'' \leftarrow T'$ and perform the following operation for each $v \in N_G(u) \cap X$.
If $v \notin V(T')$, then add he vertex v and the edge uv.
Otherwise, add the edge uv, creating a cycle C in T and remove the other edge of C incident to v. This does not decrease the number of leaves, since it only increases the number of edges incident to u, and u was already internal. □

Follow Path Lemma

Lemma 7 (Follow Path Lemma)

*Suppose $u \in B$ and $|N_G(u) \cap X| = 1$. Let $N_G(u) \cap X = \{v\}$.
If there exists a k-leaf tree extending T where u is internal, but no k-leaf tree extending T where u is a leaf, then there exists a k-leaf tree extending T where both u and v are internal.*

Follow Path Lemma

Lemma 7 (Follow Path Lemma)

Suppose $u \in B$ *and* $|N_G(u) \cap X| = 1$. *Let* $N_G(u) \cap X = \{v\}$.
If there exists a k*-leaf tree extending* T *where* u *is internal, but no* k*-leaf tree extending* T *where* u *is a leaf, then there exists a* k*-leaf tree extending* T *where both* u *and* v *are internal.*

Proof.

Suppose not, and let T' be a k-leaf tree extending T where u is internal and v is a leaf. But then, $T - v$ is a k-leaf tree as well. □

Algorithm

- Apply halting & simplification rules
- Select $u \in B$. Branch into
  - $u \in L$
  - $u \in I$. In this case, add $X \cap N_G(u)$ to B (Branching Lemma).
    - In the special case where $|X \cap N_G(u)| = 1$, denote $\{v\} = X \cap N_G(u)$, make v internal, and add $N_G(v) \cap X$ to B, continuing the same way until reaching a vertex with at least 2 neighbors in X (Follow Path Lemma).
    - In the special case where $|X \cap N_G(u)| = 0$, return No.

Algorithm

- Apply halting & simplification rules
- Select $u \in B$. Branch into
  - $u \in L$
  - $u \in I$. In this case, add $X \cap N_G(u)$ to B (Branching Lemma).
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    - In the special case where $|X \cap N_G(u)| = 0$, return No.
- In one branch, a vertex moves from B to L; in the other branch, $|B|$ increases by at least 1.

Running time analysis

- Consider the "measure" $\mu := 2k - 2|L| - |B|$
- We have that $0 \leq \mu \leq 2k$
- Branch where $u \in L$:
  - $|B|$ decreases by 1, $|L|$ increases by 1
  - μ decreases by 1
- Branch where $u \in I$.
  - u moves from B to I
  - ≥ 2 vertices move from X to B
  - μ decreases by at least 1
- Binary search tree of height $\leq \mu \leq 2k$

Result for Maximum Leaf Spanning Tree

Theorem 8 ([KLR11])

MAXIMUM LEAF SPANNING TREE *can be solved in* $O^*(4^k)$ *time.*

Current best: $O(3.188^k)$ [Zeh18]

Outline

Further Reading

- Chapter 3, *Bounded Search Trees* in [Cyg+15]
- Chapter 3, *Bounded Search Trees* in [DF13]
- Chapter 8, *Depth-Bounded Search Trees* in [Nie06]

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