

COMP4418: Knowledge Representation and Reasoning

Propositional Logic: Automating Reasoning

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Propositional Logic

- Thus far we have considered propositional logic as a knowledge representation language
- We can now write sentences in this language (syntax)
- We can also determine the truth or falsity of these sentences (semantics)
- What remains is to reason; to draw new conclusions from what we know (proof theory) and to do so using a computer to automate the process
- References:
 - ▶ Ivan Bratko, Prolog Programming for Artificial Intelligence, Addison-Wesley, 2001. (Chapter 15)
 - ▶ Stuart J. Russell and Peter Norvig, Artificial Intelligence: A Modern Approach, Prentice-Hall International, 1995. (Chapter 6)

Overview

- Normal Forms
- Resolution
- Refutation Systems
- Correctness of resolution rule — soundness and completeness revisited
- Conclusion

Motivation

If either George or Herbert wins, then both Jack and Kenneth lose
George wins

Therefore, Jack loses

$$(G \vee H) \rightarrow (\neg J \wedge \neg K)$$

G

$\neg J$

Normal Forms

- A normal form is a “standardised” version of a formula
- Common normal forms:

Negation Normal Form — negation symbols occur in front of propositional letters only (e.g., $(P \vee \neg Q) \rightarrow (P \wedge (\neg R \vee S))$)
(A literal is a propositional letter or the negation of a propositional letter.)

Conjunctive Normal Form (CNF) — a conjunct of disjunctions (e.g., $(P \vee Q \vee \neg R) \wedge (\neg S \vee \neg R)$)

Disjunctions of literals are known as clauses

Disjunctive Normal Form (DNF) — a disjunct of conjunctions (e.g., $(P \wedge Q \wedge \neg R) \vee (\neg S \wedge \neg R)$)

Negation Normal Form

- To simplify matters, let us suppose we are only dealing with formulae containing the connectives \neg , \wedge , \vee
- A (sub)formula $\phi \rightarrow \psi$ is equivalent to $\neg\phi \vee \psi$
- A (sub) formula $\phi \leftrightarrow \psi$ is equivalent to $\phi \rightarrow \psi$ and $\psi \rightarrow \phi$
- DeMorgan's laws:
 - ▶ $\neg(\phi \wedge \psi) \equiv \neg\phi \vee \neg\psi$
 - ▶ $\neg(\phi \vee \psi) \equiv \neg\phi \wedge \neg\psi$
- Double Negation: $\neg\neg P \equiv P$
- To put a formula in negation normal form, repeatedly apply De Morgan's laws and double negation
- For example, $\neg(P \vee (\neg R \wedge P)) \equiv \neg P \wedge \neg(\neg R \wedge P) \equiv \neg P \wedge (R \vee \neg P)$

Conjunctive Normal Form

- Note the following distributive identities:

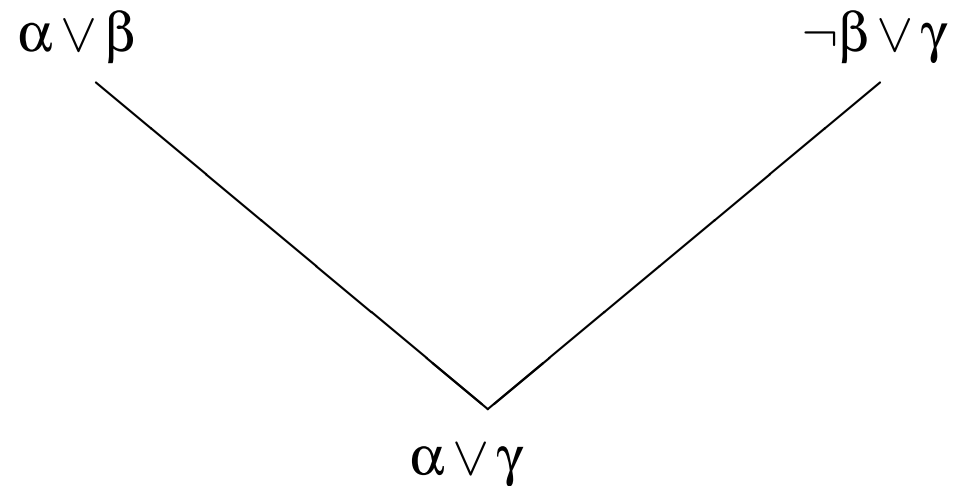
$$(\phi \wedge \psi) \vee \chi \equiv (\phi \vee \chi) \wedge (\psi \vee \chi)$$

$$(\phi \vee \psi) \wedge \chi \equiv (\phi \wedge \chi) \vee (\psi \wedge \chi)$$

- To put a formula in conjunctive normal form (CNF) firstly put the formula into negation normal form and then repeatedly apply the identities above
- For example, $R \rightarrow (P \wedge Q) \equiv (\neg R \vee P) \wedge (\neg R \vee Q)$

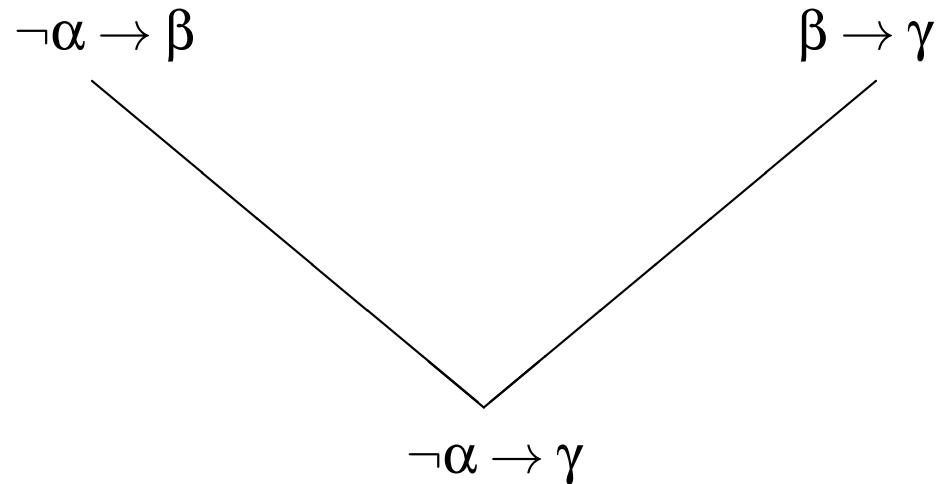
Resolution Rule

Resolution Rule:



- Where β is a literal (i.e., a propositional letter or its negation)

Resolution Rule



- Resolution is essentially equivalent to the transitivity of material implication
- In fact, it is a form of the well known cut rule in logic

Applying Resolution

- The resolution rule is sound
- What does that mean?
- How can we use the resolution rule?
 - ▶ Convert premises into CNF
 - ▶ Repeatedly apply resolution rule to the resultant clauses
 - ▶ Each clause produced can be inferred from the original premises
 - ▶ If you have a query sentence **goal**, it follows from the premises if and only if each of the clauses in $\text{CNF}(\text{goal})$ is produced by resolution
- There is a better way ...

Refutation Systems

- If we would like to prove a sentence ϕ is a theorem (i.e., $\vdash \phi$), we start with $\neg\phi$ and produce a contradiction
- A “proof by contradiction”
- Similarly, if we wish to prove $\psi_1, \dots, \psi_n \vdash \phi$, start with $\neg\phi$ and together with ψ_1, \dots, ψ_n produce a contradiction
- Resolution can be used to implement a refutation system
- Repeatedly apply resolution rule until empty clause results

Applying Resolution

- Negate conclusion (resolution is a refutation system)
- Convert premises and negated conclusion into CNF (clausal form)
- Repeatedly apply Resolution Rule, Double Negation
- If empty clause results you have a contradiction and can conclude that the conclusion follows from the premises

Resolution — Example 1

$(G \vee H) \rightarrow (\neg J \wedge \neg K), G \vdash \neg J$

$CNF[(G \vee H) \rightarrow (\neg J \wedge \neg K)] \equiv (\neg G \vee \neg J) \wedge (\neg H \vee \neg J) \wedge (\neg G \vee \neg K) \wedge (\neg H \vee \neg K)$

1. $\neg G \vee \neg J$ [Premise]
2. $\neg H \vee \neg J$ [Premise]
3. $\neg G \vee \neg K$ [Premise]
4. $\neg H \vee \neg K$ [Premise]
5. G [Premise]
6. $\neg\neg J$ [\neg Conclusion]
7. J [6. Double Negation]
8. $\neg G$ [1, 7. Resolution]
9. \square [5, 8. Resolution]

Resolution — Example 2

$$P \rightarrow \neg Q, \neg Q \rightarrow R \vdash P \rightarrow R$$

$$P \rightarrow R \equiv \neg P \vee R$$

$$\text{CNF}[\neg(\neg P \vee R)] \equiv \{\neg\neg P, \neg R\}$$

1. $\neg P \vee \neg Q$ [Premise]
2. $\neg\neg Q \vee R$ [Premise]
3. $\neg\neg P$ [\neg Conclusion]
4. $\neg R$ [\neg Conclusion]
5. P [3. Double Negation]
6. $\neg Q$ [1, 5. Resolution]
7. R [2, 6. Resolution]
8. \square [4, 7. Resolution]

Resolution — Example 3

$$\vdash ((P \vee Q) \wedge \neg P) \rightarrow Q$$

$$CNF[\neg(((P \vee Q) \wedge \neg P) \rightarrow Q)] \equiv (P \vee Q) \wedge \neg P \wedge \neg Q$$

1. $P \vee Q$ [\neg Conclusion]
2. $\neg P$ [\neg Conclusion]
3. $\neg Q$ [\neg Conclusion]
4. Q [1, 2. Resolution]
5. \square [3, 4. Resolution]

Soundness and Completeness — Recap

- An inference procedure (and hence a logic) is sound if and only if it preserves truth
- In other words \vdash is sound iff whenever $\lambda \vdash \rho$, then $\lambda \models \rho$
- A logic is complete if and only if it is capable of proving all truths
- In other words, whenever $\lambda \models \rho$, then $\lambda \vdash \rho$

Decidability

- A logic is decidable if and only if there is a mechanical procedure that, when asked $\lambda \vdash \rho$, can eventually halt and answer “yes” or halt and answer “no”
- Propositional logic is decidable

Heuristics in applying Resolution

- Clause elimination — can disregard certain types of clauses
 - ▶ Pure clauses: contain literal L where $\neg L$ doesn't appear elsewhere
 - ▶ Tautologies: clauses containing both L and $\neg L$
 - ▶ Subsumed clauses: another clause exists containing a subset of the literals
- Ordering strategies
 - ▶ Unit preference: resolve unit clauses (only one literal) first
- Many others ...

Conclusion

- We have now investigated one knowledge representation and reasoning formalism
- This means we can draw new conclusions from the knowledge we have; we can reason
- Have enough to build a knowledge-based agent
- However, propositional logic is a weak language; there are many things we can't express in it
- It cannot be used to express knowledge about objects, their properties and the relationships that exist between objects
- For this purpose we need a more expressive language: first-order logic