Foundations of Abstract Interpretation

(Week 8)

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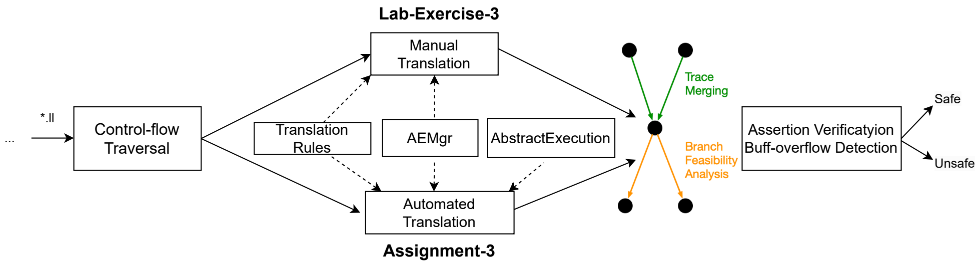
Classes in the Next Three Weeks



Outline of Today's lecture

- An Introduction to Abstract Interpretation: What and Why
- Abstract Interpretation vs Symbolic Execution
- Definitions: Abstract domains, Abstract State and Abstract Trace.
- Step-by-Step Motivating Examples.
- Widening and Narrowing to Improve Analysis Speed and Precision

Abstract Interpretation

Abstract interpretation or Abstract Execution [Cousot & Cousot, POPL'77][1], a general framework for static analysis, aims to **soundly approximate** the potential concrete values program variables may take during runtime, **based on monotonic functions over ordered sets, particularly lattices**.

Abstract Interpretation: Levels of Abstractions

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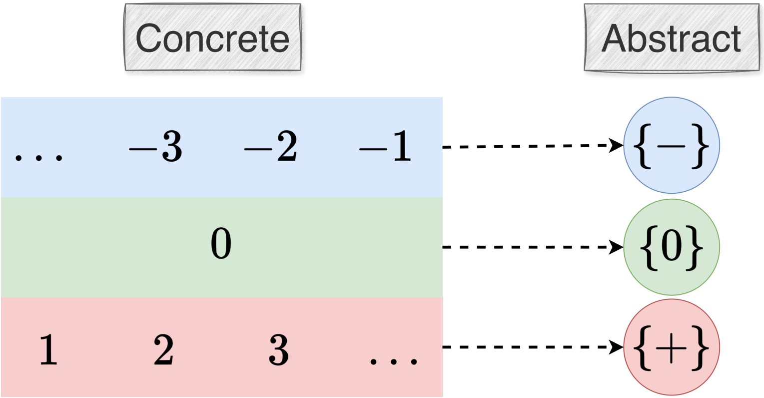

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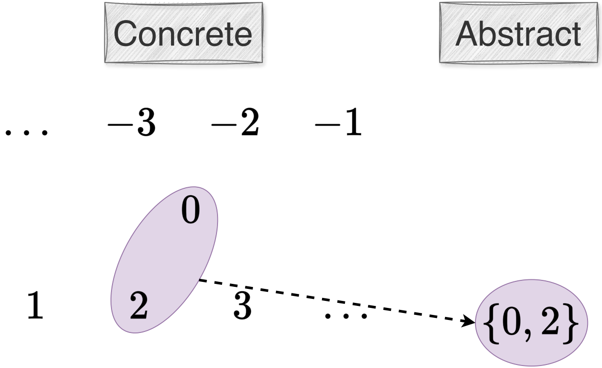

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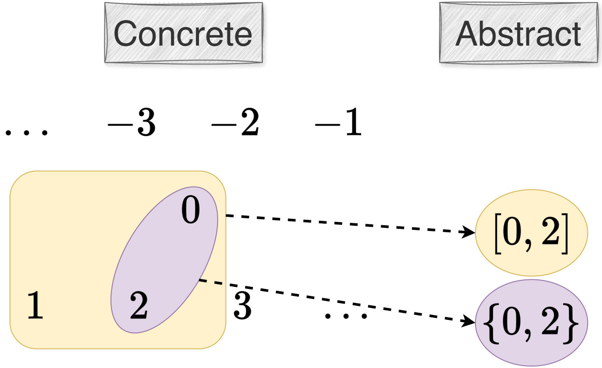

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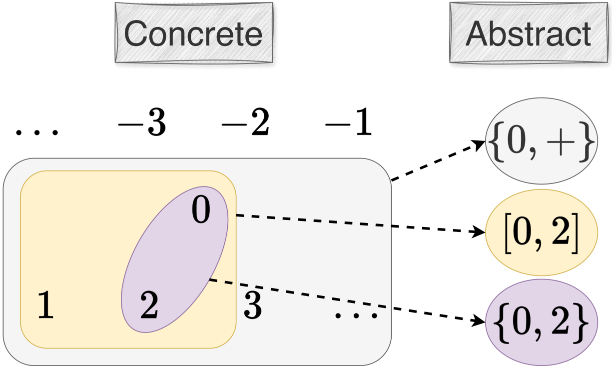

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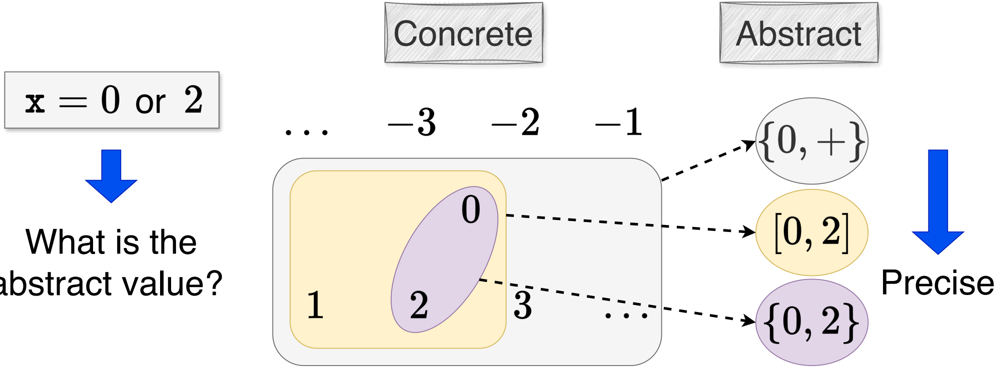

Abstract Interpretation: Applications

- **Program Optimization**: allows compilers to make safe assumptions about a program's behavior, leading to more efficient code generation.
  - **Range Analysis**: abstractly determines the loop's value range, aiding in memory optimization and eliminating redundant checks within this range.

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- **Hardware Design and Analysis:** used to verify that hardware designs meet certain specifications and to optimize the designs for better performance or lower power consumption.
  - **Analyzing Hardware Circuits:** By creating an abstract model of the circuit, it can predict how the circuit will behave under various input conditions.

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- **Hardware Design and Analysis:** used to verify that hardware designs meet certain specifications and to optimize the designs for better performance or lower power consumption.
  - **Analyzing Hardware Circuits:** By creating an abstract model of the circuit, it can predict how the circuit will behave under various input conditions.
- **Code Analysis (This Course)**: provides a systematic approach to approximate program behavior through value abstractions.
  - **Security Analysis:** crucial for early detection of bugs (e.g., assertion errors and buffer overflows), reducing debugging time and enhancing code reliability.

Abstract Interpretation: Tools

Widely used in safety-critical systems (e.g., aerospace industries) and commercial software products to enhance reliability, security, and performance.

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- **Astrée** is used to analyze and ensure the safety of software in modern aircraft, such as the Airbus A380.
- **Polyspace** is highly valued in the automotive and aerospace industries for ensuring software compliance with safety standards such as ISO 26262 for automotive software.
- **Ikos** is specialized in detecting run-time errors and numerical computation issues, making it ideal for space and aeronautics software.
- **SPARK** is used in the aerospace industry for writing and verifying safety-critical avionics software.
- **Infer** is a static analysis tool developed by Facebook to identify bugs in mobile and web applications.
- Other tools: **Frama-C, Julia Static Analyzer, BAP, Soot** and many more . . .

Abstract Interpretation vs. Symbolic Execution

Soundness

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Abstract Interpretation vs. Symbolic Execution

Soundness

- **Abstract interpretation** aims for sound results. It can conservatively approximate all possible execution paths and runtime behaviors.
- **Symbolic execution** can be unsound. It precisely explores individual yet feasible paths, facing a "path explosion" problem in large programs, and may result in under-approximation of program behaviors.

Assignment-2 vs. Assignment-3

Assignment-2

- **Delegate** the constraint solving to the **z3 SMT solver**.
- Each time, it returns **one solution with concrete values for all variables** in the search space when the solver is satisfiable.
- Per-path verification **without handling the inner parts of a loop**.

Assignment-3

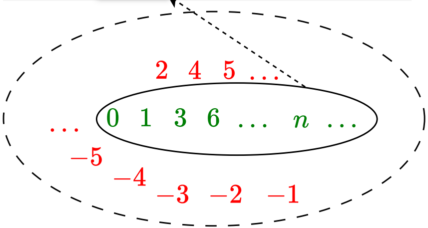
- Use **Abstract State** (`AEState`) and **Abstract Trace** (a set of `AEStates` for all ICFGNodes) to **compute and maintain abstract values** of variables.
- **Abstract all possible values** of a variable into a value **interval** (for scalars) or an **address set** (for memory addresses).
- **Approximate loop behaviors** based on widening and narrowing.

Abstract Interpretation vs. Symbolic Execution

Over-Approximation (soundness) vs. Under-Approximation (unsoundness)

```
void analyzeThis(int x) {
    int sum = 0;
    for (int i = 0; i < x; ++i) {
        sum += i;
    }
}
```

sum = ?



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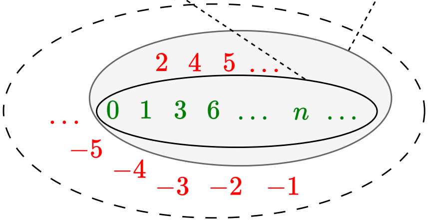

Abstract Interpretation $\{0, +\}$

Sound (include all non-negative numbers)
imprecise (may include infeasible numbers: 2, 4, 5, ...)

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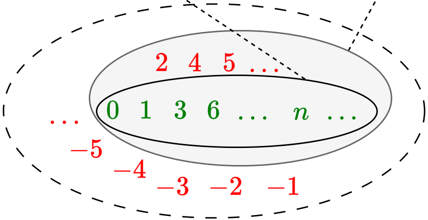

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Symbolic Execution

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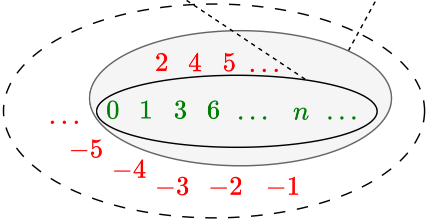

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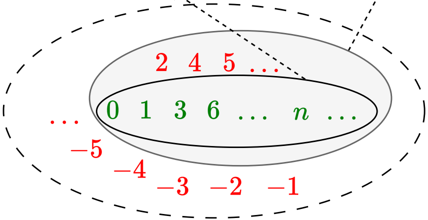

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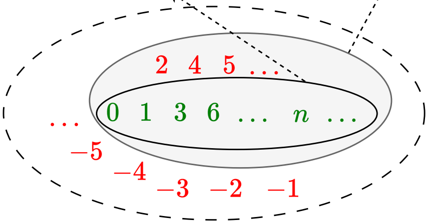

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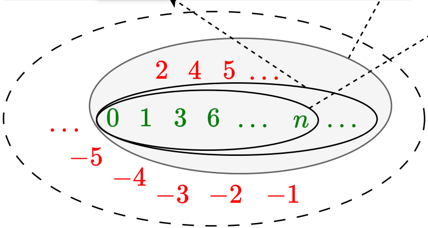

Abstract Interpretation $\{0, +\}$

Sound (include all non-negative numbers)
imprecise (may include infeasible numbers: 2, 4, 5, ...)

Symbolic Execution

Precise (only include feasible numbers: 0, 1, 3, 6, ...)
unsound (cannot cover all possible numbers)

Path	Answer
$\ell_1 \to \ell_2 \to \ell_3 \to \ell_6$	0
$\ell_1 \to \ell_2 \to \ell_3 \to \ell_4 \to \ell_5 \to \ell_6$	1
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Importance of Soundness

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- **Quality Assurance:** Crucial for critical systems where failure can have serious consequences, ensuring software behaves as intended.
- **Confidence in Maintenance:** Provides a safety net for code changes, reducing the risk of introducing new bugs.

Abstract Interpretation vs. Symbolic Execution

Termination

- **Abstract interpretation** is typically guaranteed to terminate within a finite step. Uses an abstracted, and hence more manageable, version of the state space to represent the infinite number of runtime states and paths.

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Termination

- **Abstract interpretation** is typically guaranteed to terminate within a finite step. Uses an abstracted, and hence more manageable, version of the state space to represent the infinite number of runtime states and paths.
- **Symbolic execution** may struggle with termination in complex or large-scale programs. The need to explore numerous paths in detail, especially in programs with loops and recursive calls, can lead to non-termination or impractical analysis times.

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- **Deterministic:** Ensures consistent outcomes and predictable resource use for the same input.
- **Efficiency:** Reduces computational load by using abstracted state spaces, speeding up the analysis process.
- **Coverage:** ensure that all parts of the code are analyzed, avoiding missed sections and ensuring thorough coverage for detecting issues.

Abstract Interpretation: A Code Example

```
if(cond)
   x=1;
else
   x=3;
x = ?
```

Approximation!

x	$\{+\}$

x	$[1,3]$

x	$\{1,3\}$

Abstract Interpretation: A Code Example

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Coarse-grained
but faster

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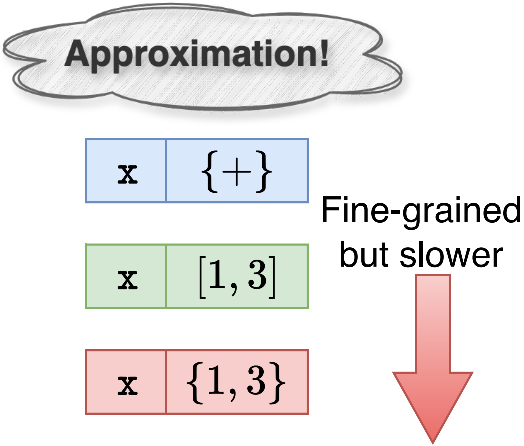


Concrete Domain and Abstract Domain: Formal Definition

Concrete Domain

- $\mathbb{S}$ denotes the set of concrete values that a program variable can have.
  - E.g., $\mathbb{S} = \mathbb{Z}$ represents the concrete values that an integer variable can have.
- A **concrete domain** $\mathbb{C}$ is the *powerset* of $\mathbb{S}$, denoted as $\mathbb{C} = \mathcal{P}(\mathbb{S})$.
  - E.g. The *powerset integer domain* is a concrete domain for integer variables.

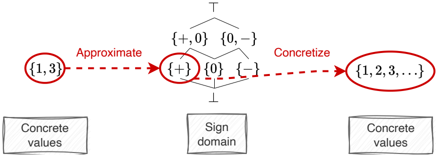
Abstract Domain

- An **abstract domain** $\mathbb{A}$ contains *abstract values* approximating a set of concrete values.
- An abstract domain is typically implemented using a **lattice** $\mathbb{L} = \langle \mathbb{A}, \sqsubseteq, \sqcap, \sqcup, \bot, \top \rangle$ structure, a set of abstract values following a **partial order**, also equipped with two binary operations.
  - $\sqsubseteq$ is a partial order relation on $\mathbb{A}$ (e.g., $\sqsubseteq$ is the subset ($\subseteq$) on a power set).
  - $\sqcap$ and $\sqcup$ are the meet and join binary operations, and $\bot$ and $\top$ are unique least and greatest elements of $\mathbb{A}$.

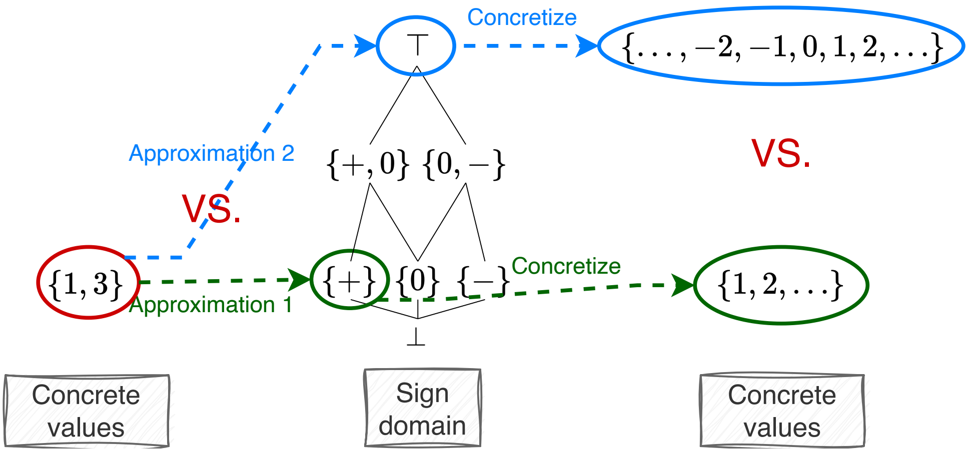
An Example: Abstract Sign Domain

An abstract domain that approximates a set of concrete values with their signs.

- Lattice is defined as $\mathbb{L} = \langle \mathcal{P}(\{-, 0, +\}), \sqsubseteq, \sqcap, \sqcup, \bot, \top \rangle$.
- **Partial order**: $a \sqsubseteq b \Leftrightarrow a \subseteq b$. E.g., $\{+\} \sqsubseteq \{0, +\} \Leftrightarrow \{+\} \subseteq \{0, +\}$.
- **Meet operator** $a \sqcap b$: returns the **greatest lower bound (GLB)** that is less than or equal to both a and b (**move downwards along the lattice**)
  - $\{+\} \sqcap \{0\} = \bot$
- **Join operator** $a \sqcup b$: returns the **least upper bound (LUB)** that is greater than or equal to both a and b (**move upwards along the lattice**)
  - $\{+\} \sqcup \{0\} = \{+, 0\}$
- **Approximation**: concrete value set $\{1, 3\}$ is **over-approximated** as $\{+\}$. After **concretization**, it is restored as $\{x \in \mathbb{Z} | x > 0\}$, a **super set** of $\{1, 3\}$.



An Example, the Best Abstraction using Sign Domain



Approximation 1 (more precise than Approximation 2) is the best abstraction!

Galois Connection

When each concrete value has a unique best abstraction, the correspondence is a **Galois connection**, which is a two-way connections between abstract domain and concrete domain using abstraction function and concretization function.

- Abstraction function $\alpha : \mathbb{C} \to \mathbb{A}$ maps a set of concrete values to its abstract ones;
- Concretization function $\gamma : \mathbb{A} \to \mathbb{C}$ maps a set of abstract values to concrete ones.

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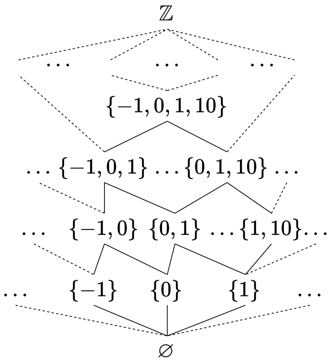
Example: Abstraction/concretization functions on sign domain

$$\gamma_{Sign}(\top) = \mathbb{Z}$$
$$\gamma_{Sign}(\{-\}) = \{x \mid x < 0\}$$
$$\gamma_{Sign}(\{+\}) = \{x \mid x > 0\}$$
$$\ldots$$

$$\alpha_{Sign}(c) = \{+\} \textit{ if } c \in \mathbb{Z}_{>0}$$
$$\alpha_{Sign}(c) = \{-\} \textit{ if } c \in \mathbb{Z}_{<0}$$
$$\alpha_{Sign}(c) = \{+, 0\} \textit{ if } c \in \mathbb{Z}_{\geq 0}$$
$$\ldots$$

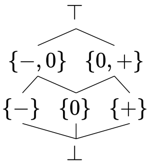
Galois Connection of Sign Domain



(a) Powerset integer domain

$$\gamma_{Sign}(\top) = \mathbb{Z}$$
$$\gamma_{Sign}(\{-\}) = \{x \mid x < 0\}$$
$$\gamma_{Sign}(\{+\}) = \{x \mid x > 0\}$$
$$\gamma_{Sign}(\{-, 0\}) = \{x \mid x \leq 0\}$$
$$\gamma_{Sign}(\{+, 0\}) = \{x \mid x \geq 0\}$$
$$\gamma_{Sign}(\{\perp\}) = \varnothing$$

$$\alpha_{Sign}(c) = \begin{cases} \{+\} \; \mathit{if}\, c \in \mathbb{Z}_{>0} \\ \{-\} \; \mathit{if}\, c \in \mathbb{Z}_{<0} \\ \{+, 0\} \; \mathit{if}\, c \in \mathbb{Z}_{\geq 0} \\ \{-, 0\} \; \mathit{if}\, c \in \mathbb{Z}_{\leq 0} \\ \perp \;\; \mathit{if}\, c = \varnothing \\ \top \; \mathit{otherwise} \end{cases}$$



(b) Sign domain

Interval Domain

The interval domain is an abstract domain that represents a set of integers that fall between two given endpoints.

- Lattice is defined as $\mathbb{L}_{interval} = \langle \mathbb{I}, \sqsubseteq, \sqcap, \sqcup, \bot, \top \rangle$, *where* $\mathbb{I} = \{[a, b] \mid a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}\} \cup \{\bot\}$.
- Partial order: $[a_1, b_1] \sqsubseteq [a_2, b_2] \Leftrightarrow a_2 \leq a_1 \wedge b_1 \leq b_2$.
  - E.g., $[0, 0], [0, 1] \in \mathbb{A}_{interval}$, satisfying $[0, 0] \sqsubseteq [0, 1]$.

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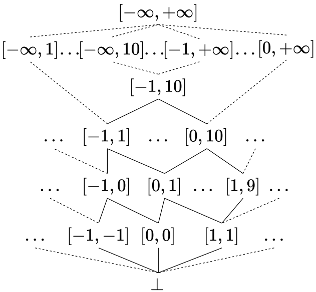


(b) Interval domain

Given $a_1 = [3, 8]$ and $a_2 = [7, 12]$.

Meet operation $a_1 \sqcap a_2$ returns the **greatest Lower Bound** (GLB):

- GLB = [7, 8], the largest range that is shared by both a_1 and a_2.

Join operation $a_1 \sqcup a_2$ returns the **Least Upper Bound** (LUB):

- LUB = [3,12], the smallest range that includes both a_1 and a_2.

LUB and GLB of lattice $\mathbb{L}_{interval}$ are $[-\infty, +\infty]$ and $\perp$ respectively.

Galois Connection between $\mathbb{C}$ and $\mathbb{A}_{interval}$

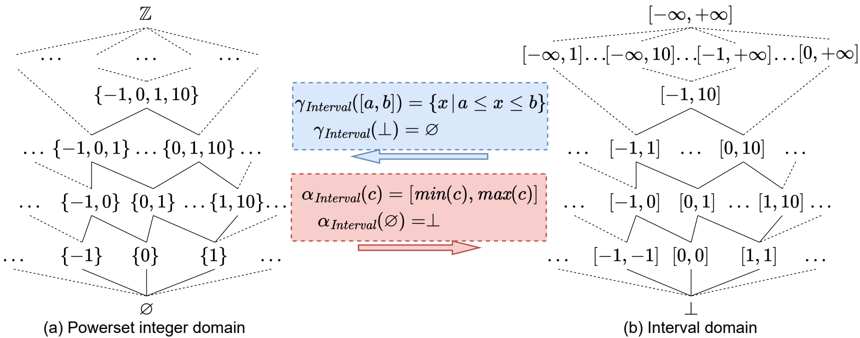


Figure: Powerset integer domain $\mathbb{C}$ and its abstraction as the interval domain $\mathbb{A}_{interval}$.

Abstract State and Abstract Trace

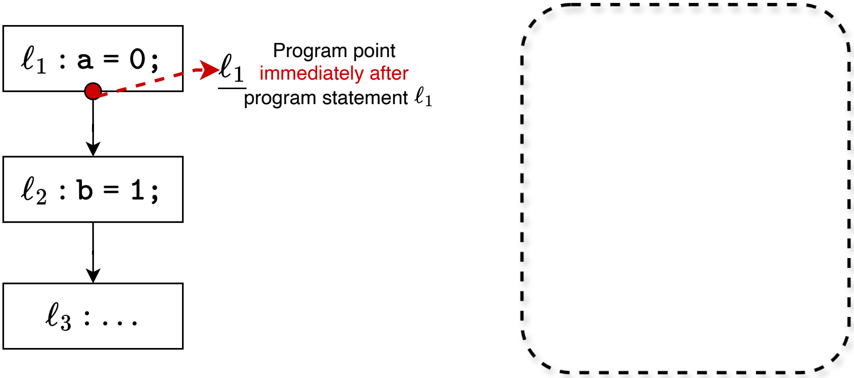
- An **abstract state** (`AEState` in Lab-3 and Assignment-3) is defined as a map $AS : \mathcal{V} \rightarrow \mathbb{A}$ associating program variables $\mathcal{V}$ with an abstract value in $\mathbb{A}$, approximating the runtime states of program variables.

Abstract State and Abstract Trace

- An **abstract state** (`AEState` in Lab-3 and Assignment-3) is defined as a map $AS : \mathcal{V} \rightarrow \mathbb{A}$ associating program variables $\mathcal{V}$ with an abstract value in $\mathbb{A}$, approximating the runtime states of program variables.
- An **abstract trace** $\sigma \in \mathbb{L} \times \mathcal{V} \rightarrow \mathbb{A}$ represents a list of abstract states before ($\overline{\ell}$) and after ($\underline{\ell}$) each program statement ℓ (`preAbsTrace` and `postAbsTrace` in Assignment-3).

	Notation	Domain
Abstract trace	σ	$\mathbb{L} \times \mathcal{V} \rightarrow \mathbb{A}_{Interval}$
Abstract state at program point $L \in \mathbb{L}$	σ_L	$\mathcal{V} \rightarrow \mathbb{A}_{Interval}$
Abstract value of x at program point $L \in \mathbb{L}$	$\sigma_L(x)$	$\mathbb{A}_{Interval}$

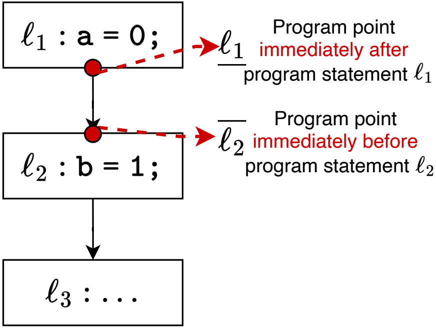
Abstract Trace : A Simple Example



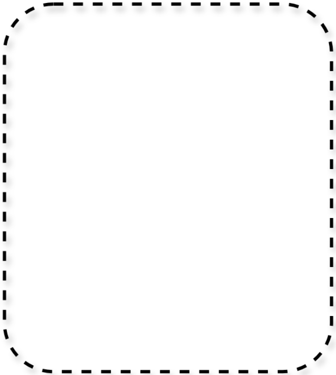
Control Flow Graph

Abstract Trace σ

Abstract Trace : A Simple Example

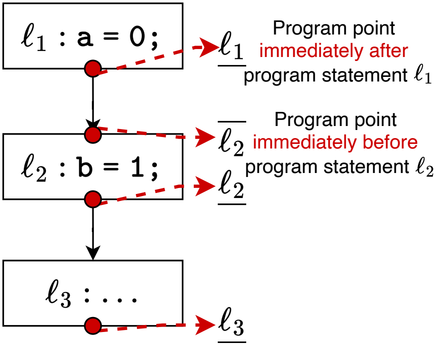


Control Flow Graph

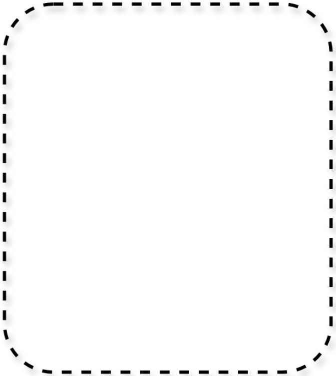


Abstract Trace σ

Abstract Trace : A Simple Example

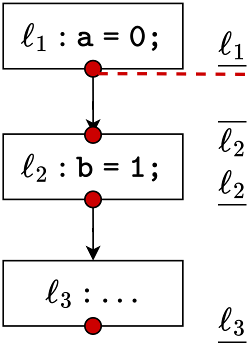


Control Flow Graph

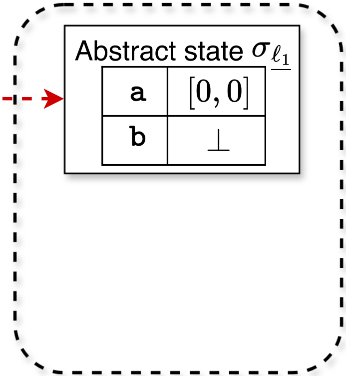


Abstract Trace σ

Abstract Trace : A Simple Example

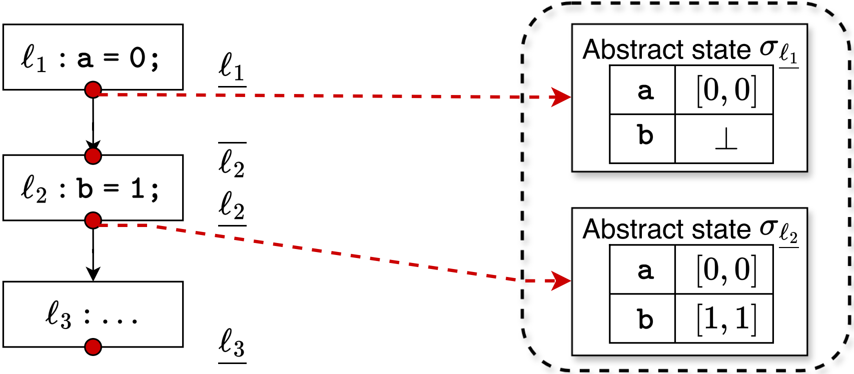


Control Flow Graph



Abstract Trace σ

Abstract Trace : A Simple Example

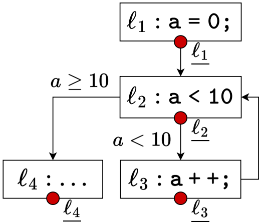


Control Flow Graph

Abstract Trace σ

Abstract Trace: Naive Fixed-Point Computation for Loops

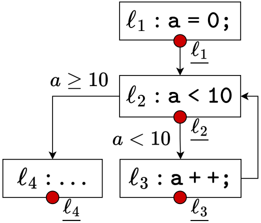
Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												

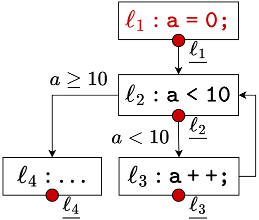


Control Flow Graph

What is the abstract state after analyzing each statement?

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

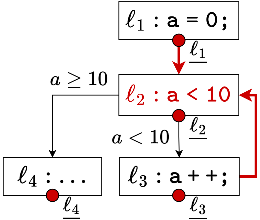
What is the abstract state after analyzing each statement?

$$\sigma_{\underline{\ell_1}}(a) := F_1() = [0, 0]$$

$F_1, \ldots, F_4$ are **transfer functions** which indicate how abstract states are updated

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

What is the abstract state after analyzing each statement?

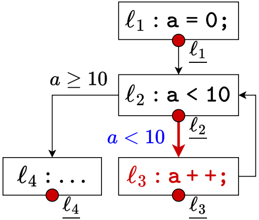
$$\sigma_{\underline{\ell_1}}(a) := F_1() = [0, 0]$$

$$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

$F_1, \ldots, F_4$ are **transfer functions** which indicate how abstract states are updated

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

What is the abstract state after analyzing each statement?

$$\sigma_{\underline{\ell_1}}(a) := F_1() = [0,0]$$

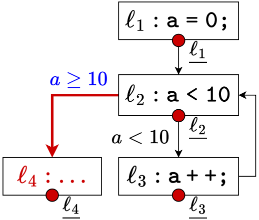
$$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

$$\sigma_{\underline{\ell_3}}(a) := F_3(\sigma_{\underline{\ell_2}}) = ([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$$

$F_1, \ldots, F_4$ are **transfer functions** which indicate how abstract states are updated

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace												
$\sigma_{\underline{\ell_1}}(a)$												
$\sigma_{\underline{\ell_2}}(a)$												
$\sigma_{\underline{\ell_3}}(a)$												
$\sigma_{\underline{\ell_4}}(a)$												



Control Flow Graph

What is the abstract state after analyzing each statement?

$$\sigma_{\underline{\ell_1}}(a) := F_1() = [0,0]$$

$$\sigma_{\underline{\ell_2}}(a) := F_2(\sigma_{\underline{\ell_1}}, \sigma_{\underline{\ell_3}}) = \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

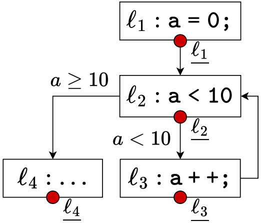
$$\sigma_{\underline{\ell_3}}(a) := F_3(\sigma_{\underline{\ell_2}}) = ([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$$

$$\sigma_{\underline{\ell_4}}(a) := F_4(\sigma_{\underline{\ell_2}}) = ([10, \infty] \sqcap \sigma_{\underline{\ell_2}}(a))$$

$F_1, \ldots, F_4$ are **transfer functions** which indicate how abstract states are updated

Abstract Trace: Naive Fixed-Point Computation for Loops

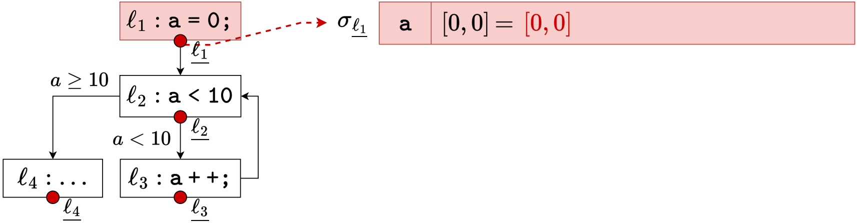
Abstract trace	Init											
$\sigma_{\underline{\ell_1}}(a)$	$\bot$											
$\sigma_{\underline{\ell_2}}(a)$	$\bot$											
$\sigma_{\underline{\ell_3}}(a)$	$\bot$											
$\sigma_{\underline{\ell_4}}(a)$	$\bot$											



Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

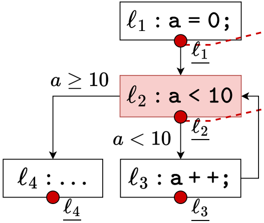
Abstract trace	Init	After analyzing ℓ_1										
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$										
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$										
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$										
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$										



Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter										
			After ℓ_2										
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$										
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$										
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$										
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$										

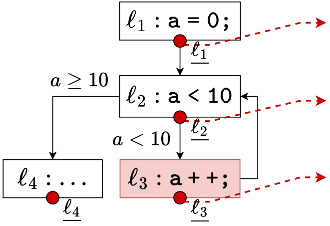


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a) = [0,0]$

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter									
			After ℓ_2	After ℓ_3								
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$								
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$								
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$								
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$								

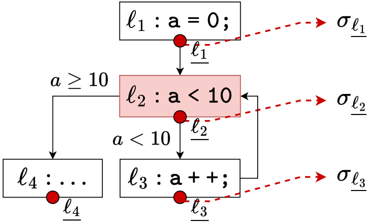


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1] = [1,1]$

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter							
			After ℓ_2	After ℓ_3	After ℓ_2							
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$							
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$							
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$							
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$							

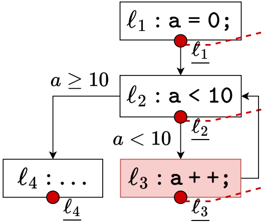


$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a) = [0,1]$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter							
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3						
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$						
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$						
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$						
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$						

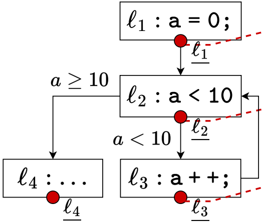


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1] = [1,2]$

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	...	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	...	$[0,10]$	$[0,10]$			
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	...	$[1,10]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$			

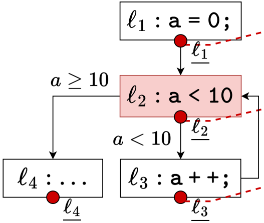


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		. . .	11^{th} loop iter		12^{nd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2		
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	. . .	$[0,0]$	$[0,0]$	$[0,0]$		
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,1]$	$[0,1]$	. . .	$[0,10]$	$[0,10]$	$[0,10]$		
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,2]$	. . .	$[1,10]$	$[1,10]$	$[1,10]$		
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	. . .	$\perp$	$\perp$	$\perp$		

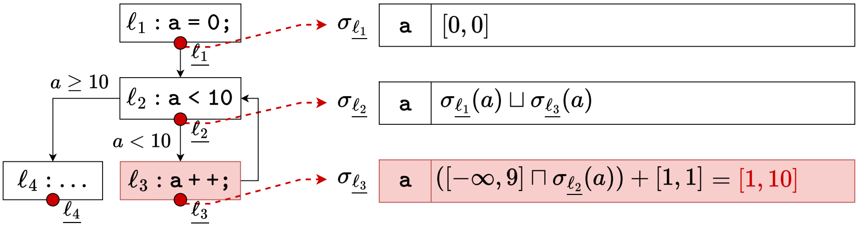


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a) = [0,10]$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1]$

Abstract Trace: Naive Fixed-Point Computation for Loops

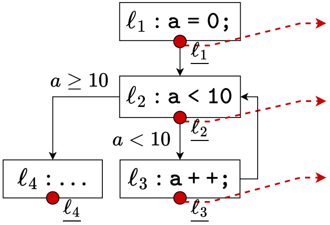
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter		12^{nd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	...	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 1]$	$[0, 1]$	...	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1, 1]$	$[1, 1]$	$[1, 2]$	...	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	



Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter		12^{nd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	...	[0, 0]	[0, 0]	[0, 0]	[0, 0]	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	[0, 1]	[0, 1]	...	[0, 10]	[0, 10]	[0, 10]	[0, 10]	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 2]	...	[1, 10]	[1, 10]	[1, 10]	[1, 10]	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	



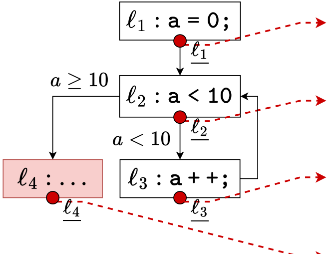
Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0, 0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1, 1]$

Fixpoint is reached!
(Abstract trace after loop round 11 = Abstract trace after loop round 12)

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		...	11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	...	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0, 0]$	$[0, 0]$	$[0, 1]$	$[0, 1]$	...	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1, 1]$	$[1, 1]$	$[1, 2]$	...	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	...	$\bot$	$\bot$	$\bot$	$\bot$	$[10, 10]$

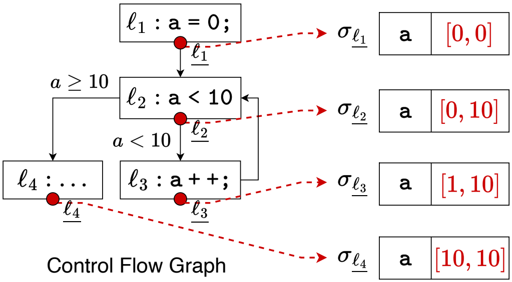


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0, 0]$
$\sigma_{\underline{\ell_2}}$	a	$\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$
$\sigma_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1, 1]$
$\sigma_{\underline{\ell_4}}$	a	$[10, \infty] \sqcap \sigma_{\underline{\ell_2}}(a) = [10, 10]$

Abstract Trace: Naive Fixed-Point Computation for Loops

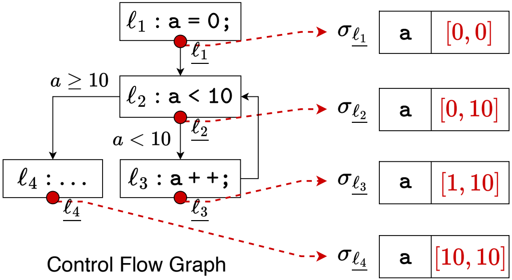
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		$\dots$	11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$\dots$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 1]$	$[0, 1]$	$\dots$	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$	$[0, 10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1, 1]$	$[1, 1]$	$[1, 2]$	$\dots$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$	$[1, 10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\dots$	$\perp$	$\perp$	$\perp$	$\perp$	$[10, 10]$



Control Flow Graph

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1th loop iter		2nd loop iter			11th loop iter		12nd loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	...	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	[0, 1]	[0, 1]	...	[0, 10]	[0, 10]	[0, 10]	[0, 10]	[0, 10]
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 2]	...	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	[10, 10]



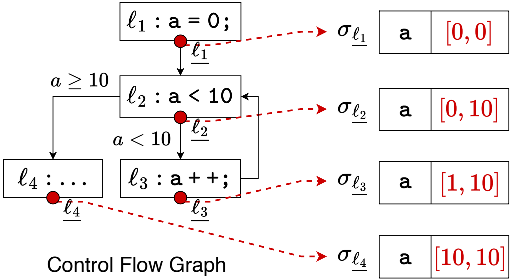
Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,1] \Rightarrow [0,2] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]}$$

12 iterations while analyzing the loop

Abstract Trace: Naive Fixed-Point Computation for Loops

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter			11^{th} loop iter		12^{nd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3		After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\ell_1}(a)$	$\perp$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	...	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	[0, 0]	[0, 0]	[0, 1]	[0, 1]	...	[0, 10]	[0, 10]	[0, 10]	[0, 10]	[0, 10]
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	[1, 1]	[1, 1]	[1, 2]	...	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	...	$\perp$	$\perp$	$\perp$	$\perp$	[10, 10]



Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,1] \Rightarrow [0,2] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]}$$

12 iterations while analyzing the loop

What if $a < 10000$?
More iterations!

Widening: Accelerating Fixed-Point Computation

Widening technique can accelerate the fixpoint computation of $\sigma_{\underline{\ell_2}}(a)$.

Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$

$$[0, 0] \Rightarrow [0, 1] \Rightarrow \ \ldots \ \Rightarrow [0, 10] \Rightarrow [0, 10]$$

Widening: Accelerating Fixed-Point Computation

Widening technique can accelerate the fixpoint computation of $\sigma_{\underline{\ell_2}}(a)$.

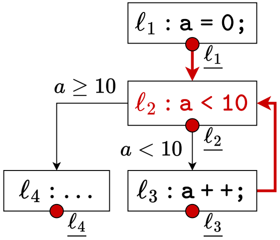
Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$

$$[0,0] \Rightarrow [0,1] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]$$

$$[0,0] \xrightarrow[\text{aggressively update } \sigma_{\underline{\ell_2}}(a)]{\text{Widening}} [0,+\infty]$$

Widening: Accelerating Fixed-Point Computation

Widening at the k^{th} iteration in the loop for analyzing ℓ_2 to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

$$\sigma_{\underline{\ell_2}}(a) := \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

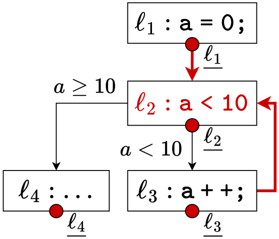
Apply widening operator ∇

$$\sigma^k_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

$\sigma^k_{\underline{\ell_2}}$ denotes the value of $\sigma_{\underline{\ell_2}}$ after the k^{th} analysis of ℓ_2, and $\sigma_{\underline{\ell_1}}$ does not have a superscription as it is updated only once and is not involved in the loop

Widening: Accelerating Fixed-Point Computation

Widening at the k^{th} iteration in the loop for analyzing ℓ_2 to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

$$\sigma_{\underline{\ell_2}}(a) := \quad \sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)$$

Apply widening operator ∇

$$\sigma^{k}_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

$\sigma^{k}_{\underline{\ell_2}}$ denotes the value of $\sigma_{\underline{\ell_2}}$ after the k^{th} analysis of ℓ_2, and $\sigma_{\underline{\ell_1}}$ does not have a superscription as it is updated only once and is not involved in the loop

What is a **Widening Operator**?

Widening Operator

The Widening Operator ($\nabla : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$) is formally defined on a poset $(\mathbb{A}, \sqsubseteq)$. ∇ on interval domain could be defined as:

$$[\ell_1, h_1]\nabla[\ell_2, h_2] = [\ell_3, h_3]$$

Widening Operator

The Widening Operator ($\nabla : \mathbb{A} \times \mathbb{A} \to \mathbb{A}$) is formally defined on a poset $(\mathbb{A}, \sqsubseteq)$. ∇ on interval domain could be defined as:

$$[\ell_1, h_1]\nabla[\ell_2, h_2] = [\ell_3, h_3]$$

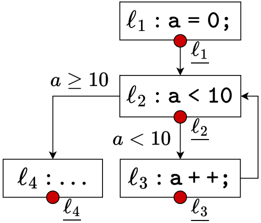
where

$$l_3 = \begin{cases} -\infty & l_2 < l_1 \\ l_1 & l_2 \geq l_1 \end{cases}, h_3 = \begin{cases} +\infty & h_2 > h_1 \\ h_1 & h_2 \leq h_1 \end{cases}$$

As a concrete example, $[0, 0]\nabla[0, 1] = [0, +\infty]$.

Widening: The Loop Example

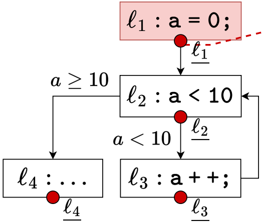
Abstract trace	Init								
$\sigma_{\underline{\ell_1}}(a)$	$\perp$								
$\sigma_{\underline{\ell_2}}(a)$	$\perp$								
$\sigma_{\underline{\ell_3}}(a)$	$\perp$								
$\sigma_{\underline{\ell_4}}(a)$	$\perp$								



Control Flow Graph

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1							
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	[0, 0]							
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$							
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$							
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$							

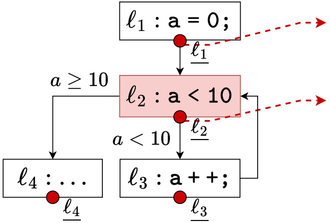


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0] = [0,\infty]$

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2						
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$						
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$						
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$						
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$						

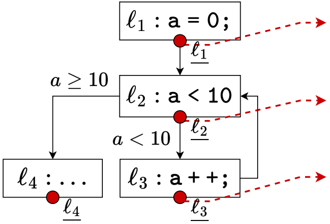


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	**a**	$[0,0]$
$\sigma_{\underline{\ell_2}}$	**a**	$(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a)) = [0,0]$

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter						
			After ℓ_2	After ℓ_3					
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$					
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$					
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$					
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$					



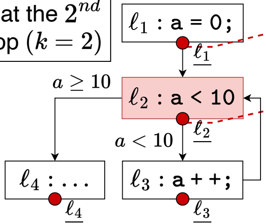
$\sigma_{\underline{\ell_1}}$	**a**	$[0,0]$
$\sigma_{\underline{\ell_2}}$	**a**	$(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma_{\underline{\ell_3}}(a))$
$\sigma_{\underline{\ell_3}}$	**a**	$([-\infty,9] \sqcap \sigma_{\underline{\ell_2}}(a)) + [1,1] = [1,1]$

Control Flow Graph

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2				
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$				
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0, 0]$	$[0, 0]$	$[0, \infty]$				
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1, 1]$	$[1, 1]$				
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$				

Start widening at the 2^{nd} iteration of loop ($k = 2$)



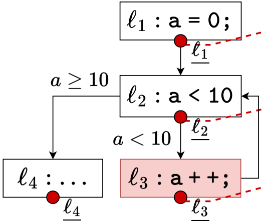
Control Flow Graph

$\sigma_{\underline{\ell_1}}$ | a | $[0, 0]$

$\sigma^2_{\underline{\ell_2}}$ | a | $\sigma^1_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^1_{\underline{\ell_3}}(a)) = [0, \infty]$

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$			
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$			



$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

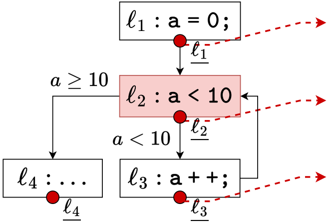
$\sigma^2_{\underline{\ell_2}}$ | a | $\sigma^1_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^1_{\underline{\ell_3}}(a))$

$\sigma^2_{\underline{\ell_3}}$ | a | $([-\infty, 9] \sqcap \sigma^2_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Control Flow Graph

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2		
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$		
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$		
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$		
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$		



Control Flow Graph

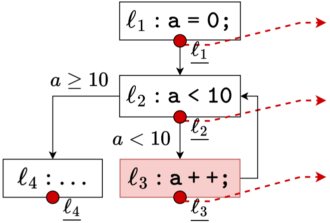
$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

$\sigma^3_{\underline{\ell_2}}$ | a | $\sigma^2_{\underline{\ell_2}}(a)\ \nabla\ (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a)) = [0,\infty]$

$\sigma^2_{\underline{\ell_3}}$ | a | $([-\infty,9] \sqcap \sigma^2_{\underline{\ell_2}}(a)) + [1,1]$

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	



Control Flow Graph

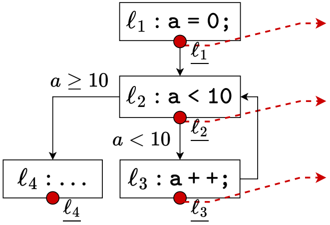
$\sigma_{\underline{\ell_1}}$: a $[0,0]$

$\sigma^3_{\underline{\ell_2}}$: a $\sigma^2_{\underline{\ell_2}}(a) \mathbin{\nabla} (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$

$\sigma^3_{\underline{\ell_3}}$: a $([-\infty,9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	



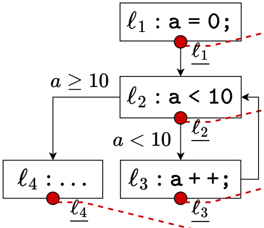
Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^3_{\underline{\ell_2}}$	a	$\sigma^2_{\underline{\ell_2}}(a) \,\nabla\, (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$
$\sigma^3_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1]$

Fixpoint is reached!
(Abstract trace after loop round 2 = Abstract trace after loop round 3)

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2	After ℓ_3	2^{nd} loop iter After ℓ_2	After ℓ_3	3^{rd} loop iter After ℓ_2	After ℓ_3	After analyzing ℓ_4
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$

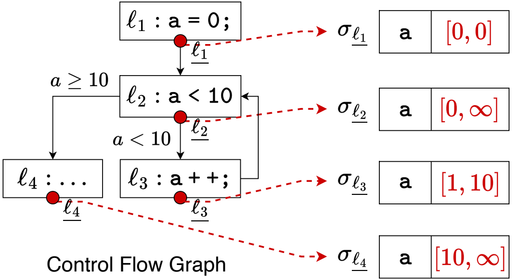


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^3_{\underline{\ell_2}}$	a	$\sigma^2_{\underline{\ell_2}}(a)\ \nabla\ (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^2_{\underline{\ell_3}}(a))$
$\sigma^3_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1]$
$\sigma_{\underline{\ell_4}}$	a	$[10,\infty] \sqcap \sigma^3_{\underline{\ell_2}}(a) = [10,\infty]$

Widening: The Loop Example

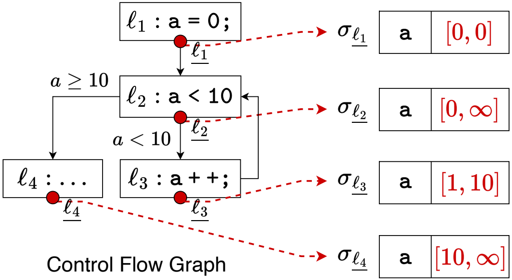
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2	After ℓ_3	2^{nd} loop iter After ℓ_2	After ℓ_3	3^{rd} loop iter After ℓ_2	After ℓ_3	After analyzing ℓ_4
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



Control Flow Graph

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



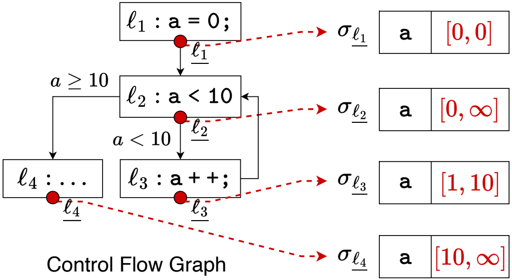
Control Flow Graph

$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty]}$

3 iterations while analyzing the loop

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



Control Flow Graph

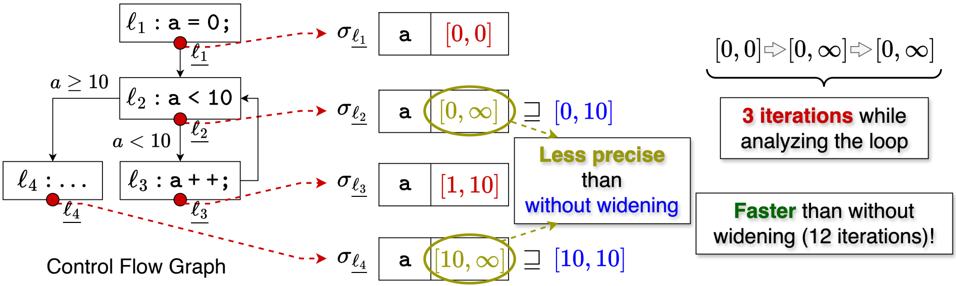
$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty]}$$

3 iterations while analyzing the loop

Faster than naive fixpoint computation (12 iterations)!

Widening: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter After ℓ_2	After ℓ_3	2^{nd} loop iter After ℓ_2	After ℓ_3	3^{rd} loop iter After ℓ_2	After ℓ_3	After analyzing ℓ_4
$\sigma_{\ell_1}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,\infty]$



Narrowing: Precision Refinement

Narrowing technique can eliminate the precision loss after a widening operation (e.g., by improving imprecise $\sigma_{\underline{\ell_2}}$ and $\sigma_{\underline{\ell_4}}$).

Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$

$$[0,0] \Rightarrow [0,1] \Rightarrow \ldots \Rightarrow [0,10] \Rightarrow [0,10]$$

$$[0,0] \xrightarrow{\text{Widening}} [0,+\infty]$$

aggressively update $\sigma_{\underline{\ell_2}}(a)$

Narrowing: Precision Refinement

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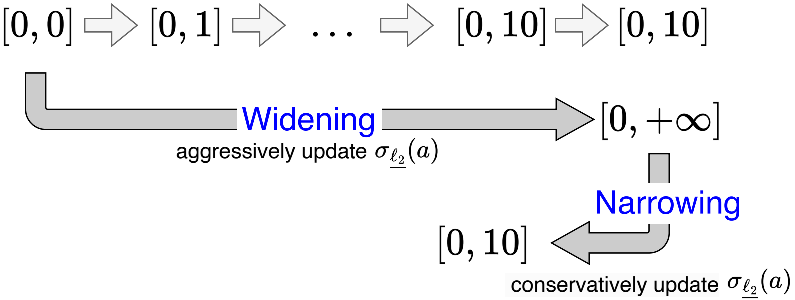
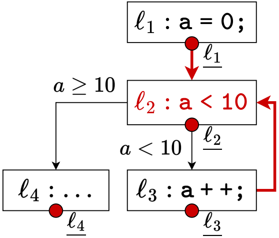
Naive fixpoint computation: value changes of $\sigma_{\underline{\ell_2}}(a)$



Narrowing: Precision Refinement

After the widening reaches a fixpoint at the k^{th} iteration when analyzing the loop, we start performing narrowing at the $(k+1)^{th}$ to update $\sigma_{\underline{\ell_2}}$.



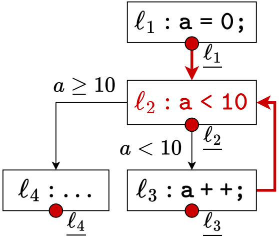
Control Flow Graph

Widening reaches a fixpoint

$$\sigma^{k}_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

Narrowing: Precision Refinement

After the widening reaches a fixpoint at the k^{th} iteration when analyzing the loop, we start performing narrowing at the $(k+1)^{th}$ to update $\sigma_{\underline{\ell_2}}$.



Control Flow Graph

Widening reaches a fixpoint

$$\sigma^{k}_{\underline{\ell_2}}(a) := \sigma^{k-1}_{\underline{\ell_2}}(a) \nabla (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^{k-1}_{\underline{\ell_3}}(a))$$

Apply narrowing operator Δ instead

Start performing narrowing

$$\sigma^{k+1}_{\underline{\ell_2}}(a) := \sigma^{k}_{\underline{\ell_2}}(a) \Delta (\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^{k}_{\underline{\ell_3}}(a))$$

What is a **Narrowing Operator**?

Narrowing Operator

The Narrowing Operator ($\Delta : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). Δ on interval domain could be defined as:

$$[l_1, h_1]\Delta[l_2, h_2] = [l_3, h_3]$$

Narrowing Operator

The Narrowing Operator ($\Delta : \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$) is formally defined on a poset ($\mathbb{A}, \sqsubseteq$). Δ on interval domain could be defined as:

$$[l_1, h_1]\Delta[l_2, h_2] = [l_3, h_3]$$

where

$$l_3 = \begin{cases} l_2 & l_1 \equiv -\infty \\ l_1 & l_1 \neq -\infty \end{cases}, h_3 = \begin{cases} h_2 & h_1 \equiv \infty \\ h_1 & h_1 \neq \infty \end{cases}$$

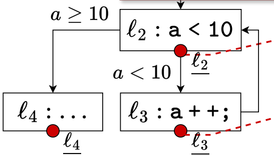
As a concrete example, $[0, \infty]\Delta[0, 10] = [0, 10]$.

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter						
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3					
$\sigma_{\ell_1}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$					
$\sigma_{\ell_2}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$					
$\sigma_{\ell_3}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$					
$\sigma_{\ell_4}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$					

Widening reaches a fixpoint at the 3^{rd} loop iteration

$), 0]$



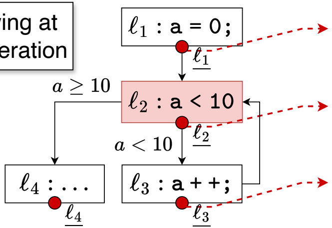
Control Flow Graph

$\sigma^3_{\ell_2}$	a	$\sigma^2_{\ell_2}(a) \nabla (\sigma_{\ell_1}(a) \sqcup \sigma^2_{\ell_3}(a))$
$\sigma^3_{\ell_3}$	a	$([-\infty, 9] \sqcap \sigma^3_{\ell_2}(a)) + [1,1]$

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2				
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$				
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$				
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$				
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$				

Start narrowing at the 4^{th} loop iteration



Control Flow Graph

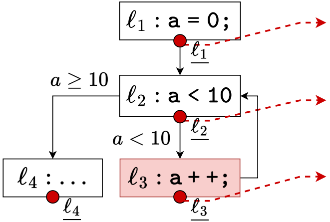
$\sigma_{\underline{\ell_1}}$ | a | $[0,0]$

$\sigma^4_{\underline{\ell_2}}$ | a | $\sigma^3_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^3_{\underline{\ell_3}}(a)) = [0,10]$

$\sigma^3_{\underline{\ell_3}}$ | a | $([-\infty,9] \sqcap \sigma^3_{\underline{\ell_2}}(a)) + [1,1]$

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter				
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3			
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$			
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$			
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$			
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$			

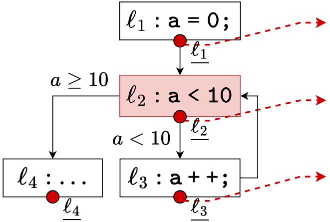


$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^4_{\underline{\ell_2}}$	a	$\sigma^3_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^3_{\underline{\ell_3}}(a))$
$\sigma^4_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^4_{\underline{\ell_2}}(a)) + [1,1] = [1,10]$

Control Flow Graph

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2		
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$		
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$		
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$		
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$		

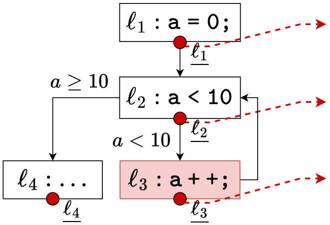


Control Flow Graph

	a	$[0,0]$
$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a)) = [0,10]$
$\sigma^4_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^4_{\underline{\ell_2}}(a)) + [1,1]$

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\bot$	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	[0, 0]	
$\sigma_{\underline{\ell_2}}(a)$	$\bot$	$\bot$	[0, 0]	[0, 0]	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	$[0, \infty]$	[0, 10]	[0, 10]	[0, 10]	[0, 10]	
$\sigma_{\underline{\ell_3}}(a)$	$\bot$	$\bot$	$\bot$	[1, 1]	[1, 1]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	[1, 10]	
$\sigma_{\underline{\ell_4}}(a)$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	

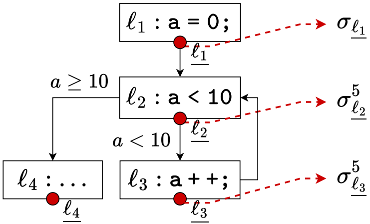


$\sigma_{\underline{\ell_1}}$	a	$[0, 0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a))$
$\sigma^5_{\underline{\ell_3}}$	a	$([-\infty, 9] \sqcap \sigma^5_{\underline{\ell_2}}(a)) + [1, 1] = [1, 10]$

Control Flow Graph

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10$	$[0,10]$	
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10$	$[1,10]$	
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	



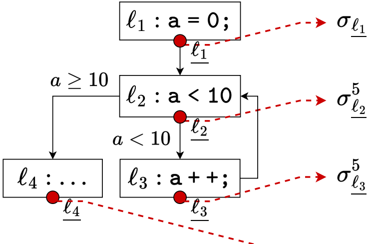
Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a))$
$\sigma^5_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^5_{\underline{\ell_2}}(a)) + [1,1]$

Fixpoint is reached!
(Abstract trace after loop round 4 = Abstract trace after loop round 5)

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$

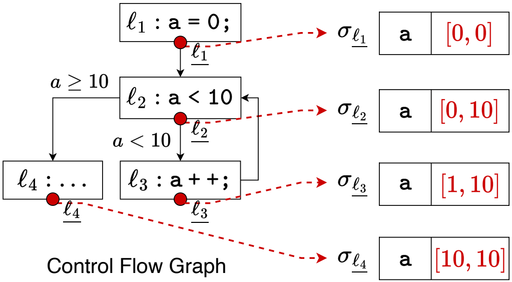


Control Flow Graph

$\sigma_{\underline{\ell_1}}$	a	$[0,0]$
$\sigma^5_{\underline{\ell_2}}$	a	$\sigma^4_{\underline{\ell_2}}(a)\,\Delta(\sigma_{\underline{\ell_1}}(a) \sqcup \sigma^4_{\underline{\ell_3}}(a))$
$\sigma^5_{\underline{\ell_3}}$	a	$([-\infty,9] \sqcap \sigma^5_{\underline{\ell_2}}(a)) + [1,1]$
$\sigma_{\underline{\ell_4}}$	a	$[10,\infty] \sqcap \sigma^5_{\underline{\ell_2}}(a) = [10,10]$

Narrowing: The Loop Example

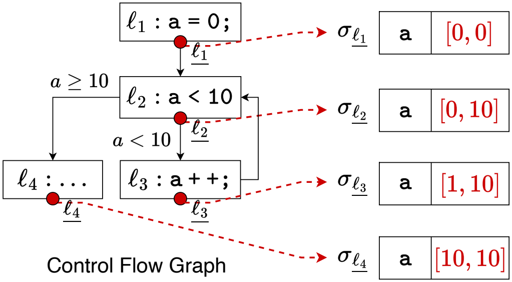
Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



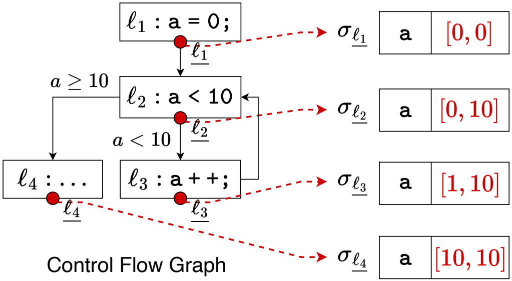
Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty] \Rightarrow [0,10] \Rightarrow [0,10]}$$

5 iterations while analyzing the loop

Narrowing: The Loop Example

Abstract trace	Init	After analyzing ℓ_1	1^{th} loop iter		2^{nd} loop iter		3^{rd} loop iter		4^{th} loop iter		5^{th} loop iter		After analyzing ℓ_4
			After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	After ℓ_2	After ℓ_3	
$\sigma_{\underline{\ell_1}}(a)$	$\perp$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$	$[0,0]$
$\sigma_{\underline{\ell_2}}(a)$	$\perp$	$\perp$	$[0,0]$	$[0,0]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,\infty]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$	$[0,10]$
$\sigma_{\underline{\ell_3}}(a)$	$\perp$	$\perp$	$\perp$	$[1,1]$	$[1,1]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$	$[1,10]$
$\sigma_{\underline{\ell_4}}(a)$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	$[10,10]$



Control Flow Graph

$$\underbrace{[0,0] \Rightarrow [0,\infty] \Rightarrow [0,\infty] \Rightarrow [0,10] \Rightarrow [0,10]}$$

5 iterations while analyzing the loop

Faster!

Precise!