4. Basics of Parameterized Complexity
COMP6741: Parameterized and Exact Computation

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Outline

1 Introduction
   - Vertex Cover
   - Coloring
   - Clique
   - $\Delta$-Clique

2 Basic Definitions

3 Further Reading
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A vertex cover in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every edge of $G$ has at least one endpoint in $S$.

**Vertex Cover**

Input: A graph $G = (V, E)$ and an integer $k$

Parameter: $k$

Question: Does $G$ have a vertex cover of size $k$?
Algorithms for Vertex Cover

- brute-force: $O^*(2^n)$
- brute-force: $O^*(n^k)$
- vc1: $O^*(2^k)$ (cf. Lecture 1)
- vc2: $O^*(1.4656^k)$ (cf. Lecture 1)
- fastest known: $O(1.2738^k + k \cdot n)$ [Chen, Kanj, Xia, 2010]
Running times in practice

\( n = 1000 \) vertices,
\( k = 20 \) parameter

\[
\begin{array}{c|c|c}
\text{Theoretical} & \text{Running Time} & \text{Real} \\
\hline
2^n & 1.07 \cdot 10^{301} & 4.941 \cdot 10^{282} \text{ years} \\
n^k & 10^{60} & 4.611 \cdot 10^{41} \text{ years} \\
2^k \cdot n & 1.05 \cdot 10^9 & 15.26 \text{ milliseconds} \\
1.4656^k \cdot n & 2.10 \cdot 10^6 & 0.31 \text{ milliseconds} \\
1.2738^k + k \cdot n & 2.02 \cdot 10^4 & 0.0003 \text{ milliseconds} \\
\end{array}
\]

Notes:
- We assume that \( 2^{36} \) instructions are carried out per second.
- The Big Bang happened roughly \( 13.5 \cdot 10^9 \) years ago.
Goal of Parameterized Complexity

Confine the combinatorial explosion to a parameter $k$.

(1) Which problem–parameter combinations are fixed-parameter tractable (FPT)? In other words, for which problem–parameter combinations are there algorithms with running times of the form

$$f(k) \cdot n^{O(1)},$$

where the $f$ is a computable function independent of the input size $n$?

(2) How small can we make the $f(k)$?
Examples of Parameters

A Parameterized Problem

Input: an instance of the problem
Parameter: a parameter
Question: a Yes–No question about the instance and the parameter

- A parameter can be
  - solution size
  - input size (trivial parameterization)
  - related to the structure of the input (maximum degree, treewidth, branchwidth, genus, ...)
  - combinations of parameters
  - etc.
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A $k$-coloring of a graph $G = (V, E)$ is a function $f : V \rightarrow \{1, 2, \ldots, k\}$ assigning colors to $V$ such that no two adjacent vertices receive the same color.

**COLORING**

Input: Graph $G$, integer $k$
Parameter: $k$
Question: Does $G$ have a $k$-coloring?

Brute-force: $O^*(k^n)$, where $n = |V(G)|$.
Inclusion-Exclusion: $O^*(2^n)$.
FPT?
Known: \textsc{Coloring} is \textsf{NP}-complete when $k = 3$

Suppose there was a $O^*(f(k))$-time algorithm for \textsc{Coloring}
  
  Then, $3$-\textsc{Coloring} can be solved in $O^*(f(3)) \subseteq O^*(1)$ time

Therefore, $P = \textsf{NP}$

Therefore, \textsc{Coloring} is not \textsf{FPT} unless $P = \textsf{NP}$
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A clique in a graph $G = (V, E)$ is a subset of its vertices $S \subseteq V$ such that every two vertices from $S$ are adjacent in $G$.

**Clique**

Input: Graph $G = (V, E)$, integer $k$
Parameter: $k$
Question: Does $G$ have a clique of size $k$?

Is Clique NP-complete when $k$ is a fixed constant? Is it FPT?
Algorithm for Clique

- For each subset \( S \subseteq V \) of size \( k \), check whether all vertices of \( S \) are adjacent
- Running time: \( O^*(\binom{n}{k}) \subseteq O^*(n^k) \)
- When \( k \in O(1) \), this is polynomial
- But: we do not currently know an \textbf{FPT} algorithm for \textbf{CLIQUE}
- Since \textbf{CLIQUE} is \( W[1] \)-hard, we believe it is not \textbf{FPT}. (See lecture on \( W \)-hardness.)
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A different parameter for Clique

$\Delta$-CLIQUE

Input: Graph $G = (V, E)$, integer $k$
Parameter: $\Delta(G)$, i.e., the maximum degree of $G$
Question: Does $G$ have a clique of size $k$?

Is $\Delta$-CLIQUE FPT?
Algorithm for $\Delta$-Clique

- If $k = 0$, answer **Yes**.
- If $k > \Delta + 1$, answer **No**.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$.  

Running time: $O^*((\Delta + 1)^k) \subseteq O^*((\Delta + 1)^\Delta)$. (FPT for parameter $\Delta$)
Algorithm for $\Delta$-Clique

- If $k = 0$, answer **Yes**.
- If $k > \Delta + 1$, answer **No**.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$.
  - // For each $v \in V$, we will check whether $G$ has a clique of size $k$ containing $v$.
  - // Note that for a clique $S$ containing $v$, we have $S \subseteq N_G[v]$.
  - For each $v \in V$, check for each vertex subset $S \subseteq N_G[v]$ of size $k$ whether $S$ is a clique in $G$. 

Running time: $O^*((\Delta + 1)^k) \subseteq O^*((\Delta + 1)^{\Delta})$. (FPT for parameter $\Delta$)
Algorithm for $\Delta$-Clique

- If $k = 0$, answer **Yes**.
- If $k > \Delta + 1$, answer **No**.
- Otherwise,
  - // A clique of size $k$ contains at least one vertex $v$. We try all possibilities for $v$.
  - // For each $v \in V$, we will check whether $G$ has a clique of size $k$ containing $v$.
  - // Note that for a clique $S$ containing $v$, we have $S \subseteq N_G[v]$.
  - For each $v \in V$, check for each vertex subset $S \subseteq N_G[v]$ of size $k$ whether $S$ is a clique in $G$.
- Running time: $O^*((\Delta + 1)^k) \subseteq O^*((\Delta + 1)^\Delta)$. (**FPT** for parameter $\Delta$)
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Main Parameterized Complexity Classes

\( n \): instance size  
\( k \): parameter

**P**: class of problems that can be solved in \( n^{O(1)} \) time  
**FPT**: class of parameterized problems that can be solved in \( f(k) \cdot n^{O(1)} \) time  
**XP**: class of parameterized problems that can be solved in \( f(k) \cdot n^{g(k)} \) time  
("polynomial when \( k \) is a constant")

\[ P \subseteq \text{FPT} \subseteq W[1] \subseteq W[2] \cdots \subseteq W[P] \subseteq \text{XP} \]

**Known**: If \( \text{FPT} = W[1] \), then the Exponential Time Hypothesis fails, i.e. 3-SAT can be solved in \( 2^{o(n)} \) time, where \( n \) is the number of variables.

**Note**: We assume that \( f \) is computable and non-decreasing.
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