5b. Branching algorithms
COMP6741: Parameterized and Exact Computation

Serge Gaspers
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1 Branching algorithms

Branching Algorithm

• Selection: Select a local configuration of the problem instance
• Recursion: Recursively solve subinstances
• Combination: Compute a solution of the instance based on the solutions of the subinstances

• Halting rule: 0 recursive calls
• Simplification rule: 1 recursive call
• Branching rule: $\geq 2$ recursive calls

Example: Our first Vertex Cover algorithm

Algorithm $vc1(G, k)$:

1 if $E = \emptyset$ then // all edges are covered
2 \hspace{1em} return Yes
3 else if $k \leq 0$ then // we cannot select any vertex
4 \hspace{1em} return No
5 else
6 \hspace{1em} Select an edge $uv \in E$;
7 \hspace{1em} return $vc1(G - u, k - 1) \lor vc1(G - v, k - 1)$
2 Running time analysis

Search trees

Recall: A search tree models the recursive calls of an algorithm. For a \( b \)-way branching where the parameter \( k \) decreases by \( a \) at each recursive call, the number of nodes is at most \( b^{k/a} \cdot (k/a + 1) \).

\[
\begin{align*}
&k \\
&k - a \\
&k - 2a \\
\hdots \\
&k - 2a
\end{align*}
\leq b^{k/a}
\leq \frac{k}{a} + 1
\leq \frac{k}{a} + 1
\]

If \( k/a \) and \( b \) are upper bounded by a function of \( k \), and the time spent at each node is FPT (typically, polynomial), then we get an FPT running time.

3 Feedback Vertex Set

A feedback vertex set of a multigraph \( G = (V, E) \) is a set of vertices \( S \subseteq V \) such that \( G - S \) is acyclic.

<table>
<thead>
<tr>
<th>Feedback Vertex Set</th>
</tr>
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<tbody>
<tr>
<td>Input: Multigraph ( G = (V, E) ), integer ( k )</td>
</tr>
<tr>
<td>Parameter: ( k )</td>
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<tr>
<td>Question: Does ( G ) have a feedback vertex set of size at most ( k )?</td>
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Simplification Rules

We apply the first applicable simplification rule.

(Loop)
If \( G \) has a loop \( vv \in E \), then set \( G \leftarrow G - v \) and \( k \leftarrow k - 1 \).

(Multiedge)
If \( E \) contains an edge \( uv \) more than twice, remove all but two copies of \( uv \).

(Degree-1)
If \( \exists v \in V \) with \( d_G(v) \leq 1 \), then set \( G \leftarrow G - v \).

(Budget-exceeded)
If \( k < 0 \), then return No.

(Degree-2)
If \( \exists v \in V \) with \( d_G(v) = 2 \), then denote \( N_G(v) = \{u, w\} \) and set \( G \leftarrow G' = (V \setminus \{v\}, (E \setminus \{vu, vw\}) \cup \{uw\}) \).

Lemma 1. (Degree-2) is sound.

\[ ^1 \text{A simplification rule is applicable if it modifies the instance.} \]
Proof. Suppose $S$ is a feedback vertex set of $G$ of size at most $k$. Let

$$S' = \begin{cases} S & \text{if } v \not\in S \\ (S \setminus \{v\}) \cup \{u\} & \text{if } v \in S. \end{cases}$$

Now, $|S'| \leq k$ and $S'$ is a feedback vertex set of $G'$ since every cycle in $G'$ corresponds to a cycle in $G$, with, possibly, the edge $uw$ replaced by the path $(u,v,w)$. Suppose $S'$ is a feedback vertex set of $G'$ of size at most $k$. Then, $S'$ is also a feedback vertex set of $G$. \qed

Remaining issues

- A select–discard branching decreases $k$ in only one branch
- One could branch on all the vertices of a cycle, but the length of a shortest cycle might not be bounded by any function of $k$

Idea:

- An acyclic graph has average degree $< 2$
- After applying simplification rules, $G$ has average degree $\geq 3$
- The selected feedback vertex set needs to be incident to many edges
- Does a feedback vertex set of size at most $k$ contain at least one vertex among the $f(k)$ vertices of highest degree?

The fvs needs to be incident to many edges

Lemma 2. If $S$ is a feedback vertex set of $G = (V,E)$, then

$$\sum_{v \in S} (d_G(v) - 1) \geq |E| - |V| + 1$$

Proof. Since $F = G - S$ is acyclic, $|E(F)| \leq |V| - |S| - 1$. Since every edge in $E \setminus E(F)$ is incident with a vertex of $S$, we have

$$|E| = |E| - |E(F)| + |E(F)| \leq \left( \sum_{v \in S} d_G(v) \right) + (|V| - |S| - 1) = \left( \sum_{v \in S} (d_G(v) - 1) \right) + |V| - 1. \quad \square$$

The fvs needs to contain a high-degree vertex

Lemma 3. Let $G$ be a graph with minimum degree at least 3 and let $H$ denote a set of $3k$ vertices of highest degree in $G$. Every feedback vertex set of $G$ of size at most $k$ contains at least one vertex of $H$.

Proof. Suppose not. Let $S$ be a feedback vertex set with $|S| \leq k$ and $S \cap H = \emptyset$. Then,

$$2|E| - |V| = \sum_{v \in V} (d_G(v) - 1) = \sum_{v \in H} (d_G(v) - 1) + \sum_{v \in V \setminus H} (d_G(v) - 1) \geq 3 \cdot (\sum_{v \in S} (d_G(v) - 1)) = \sum_{v \in S} (d_G(v) - 1) \geq 4 \cdot (|E| - |V| + 1) \implies 3|V| \geq 2|E| + 4.$$

But this contradicts the fact that every vertex of $G$ has degree at least 3. \qed
Algorithm for Feedback Vertex Set

**Theorem 4.** **Feedback Vertex Set** can be solved in $O^*((3k)^k)$ time.

**Proof (sketch).**
- Exhustively apply the simplification rules.
- The branching rule computes $H$ of size $3k$, and branches into subproblems $(G - v, k - 1)$ for each $v \in H$.

Current best: $O^*(3.591^k)$ deterministic [Kociumaka, Pilipczuk, 2014], $O^*(3^k)$ time randomized [Cygan et al., 2011]

4 Maximum Leaf Spanning Tree

A *leaf* of a tree is a vertex with degree 1. A *spanning tree* in a graph $G = (V, E)$ is a subgraph of $G$ that is a tree and has $|V|$ vertices.

**Property**

A $k$-leaf tree in $G$ is a subgraph of $G$ that is a tree with at least $k$ leaves. A $k$-leaf spanning tree in $G$ is a spanning tree in $G$ with at least $k$ leaves.

**Lemma 5.** Let $G = (V, E)$ be a connected graph. $G$ has a $k$-leaf tree $\iff$ $G$ has a $k$-leaf spanning tree.

**Proof.** ($\Leftarrow$): trivial

($\Rightarrow$): Let $T$ be a $k$-leaf tree in $G$. By induction on $x := |V| - |V(T)|$, we will show that $T$ can be extended to a $k$-leaf spanning tree in $G$.

Base case: $x = 0 \swarrow$.

Induction: $x > 0$, and assume the claim is true for all $x' < x$. Choose $uv \in E$ such that $u \in V(T)$ and $v \notin V(T)$. Since $T' := (V(T) \cup \{v\}, E(T) \cup \{uv\})$ has $\geq k$ leaves and $< x$ external vertices, it can be extended to a $k$-leaf spanning tree in $G$ by the induction hypothesis.

**Strategy**

- The branching algorithm will check whether $G$ has a $k$-leaf tree.
- A tree with $\geq 3$ vertices has at least one internal (= non-leaf) vertex.
- “Guess” an internal vertex $r$, i.e., do a $|V|$-way branching fixing an initial internal vertex $r$.
- In any branch, the algorithm has computed
  - $T$ - a tree in $G$
  - $I$ - the internal vertices of $T$, with $r \in I$
  - $B$ - a subset of the leaves of $T$ where $T$ may be extended: the boundary set
  - $L$ - the remaining leaves of $T$
  - $X$ - the external vertices $V \setminus V(T)$
- The question is whether $T$ can be extended to a $k$-leaf tree where all the vertices in $L$ are leaves.
Simplification Rules

Apply the first applicable simplification rule:

(Halt-Yes)
If \(|L| + |B| \geq k\), then return Yes.

(Halt-No)
If \(|B| = 0\), then return No.

(Non-extendable)
If \(\exists v \in B\) with \(N_G(v) \cap X = \emptyset\), then move \(v\) to \(L\).

Branching Lemma

Lemma 6 (Branching Lemma). Suppose \(u \in B\) and there exists a k-leaf tree \(T'\) extending \(T\) where \(u\) is an internal vertex. Then, there exists a k-leaf tree \(T''\) extending \((V(T) \cup N_G(u), E(T) \cup \{uv : v \in N_G(u) \cap X\})\).

Proof. Start from \(T'' \leftarrow T'\) and perform the following operation for each \(v \in N_G(u) \cap X\).

If \(v \notin V(T')\), then add the vertex \(v\) and the edge \(uv\). Otherwise, add the edge \(uv\), creating a cycle \(C\) in \(T\) and remove the other edge of \(C\) incident to \(v\). This does not decrease the number of leaves, since it only increases the number of edges incident to \(u\), and \(u\) was already internal. \(\square\)

Follow Path Lemma

Lemma 7 (Follow Path Lemma). Suppose \(u \in B\) and \(|N_G(u) \cap X| = 1\). Let \(N_G(u) \cap X = \{v\}\). If there exists a k-leaf tree extending \(T\) where \(u\) is internal, but no k-leaf tree extending \(T\) where \(u\) is a leaf, then there exists a k-leaf tree extending \(T\) where both \(u\) and \(v\) are internal.

Proof. Suppose not, and let \(T'\) be a k-leaf tree extending \(T\) where \(u\) is internal and \(v\) is a leaf. But then, \(T - v\) is a k-leaf tree as well. \(\square\)

Algorithm

- Apply simplification rules
- Select \(u \in B\). Branch into
  - \(u \in L\)
  - \(u \in I\). In this case, add \(X \cap N_G(u)\) to \(B\) (Branching Lemma). In the special case where \(|X \cap N_G(u)| = 1\), denote \(\{v\} = X \cap N_G(u)\), make \(v\) internal, and add \(N_G(v) \cap X\) to \(B\), continuing the same way until reaching a vertex with at least 2 neighbors in \(X\) (Follow Path Lemma).
  - In one branch, a vertex moves from \(B\) to \(L\); in the other branch, \(|B|\) increases by at least 1.

Running time analysis

- Measure \(\mu := 2k - 2|L| - |B| \geq 0\).
- Branch where \(u \in L\):
  - \(|B|\) decreases by 1, \(|L|\) increases by 1
  - \(\mu\) decreases by 1
- Branch where \(u \in I\):
  - \(u\) moves from \(B\) to \(I\)
  - \(\geq 2\) vertices move from \(X\) to \(B\)
  - \(\mu\) decreases by at least 1
- Binary search tree
- Height \(\leq \mu \leq 2k\)
Result for Maximum Leaf Spanning Tree

**Theorem 8** ([Kneis, Langer, Rossmanith, 2011]). **Maximum Leaf Spanning Tree** can be solved in $O^*(4^k)$ time.

Current best: $O(3.188^k)$ [Zehavi, 2018]

## 5 Further Reading

