Exercise 1. Arrange the following functions in increasing order of growth: \( n \log n; (\log n)^n; 2^n; 2^{2^n}; 2^{n^2}; n!; 1.01^n; 50^n; 2^{n/2}; 2\sqrt{n} \).

Solution. By increasing order of growth: \( n \log n; 2^{\sqrt{n}}; 1.01^n; 2^{n/2}; 2^n; (\log n)^n; n!; 2^{n^2}; 2^{2^n} \).

Exercise 2. Show that VERTEX COVER can be solved in polynomial time for graphs of maximum degree at most 2.

Solution. First, we make an observation about the structure of such graphs.

Observation 1. A graph of maximum degree at most 2 is a disjoint union of paths and cycles \(^1\).

Denote by \( \tau(G) \) the vertex cover number of \( G \), i.e., the size of a smallest vertex cover of \( G \). By the following observation, the vertex cover number of \( G \) equals the sum of the vertex cover numbers of the connected components of \( G \).

Observation 2. A (smallest) vertex cover of a graph is the union of (smallest) vertex covers of each of its connected components.

Now, it suffices to optimally solve VERTEX COVER on these two types of graphs.

Lemma 3. For a path \( P_k \) on \( k \geq 1 \) vertices, \( \tau(P_k) = \lceil (k-1)/2 \rceil \).

Proof. The proof is by induction on \( k \).

For the base cases \( k = 1 \) and \( k = 2 \), note that an edgeless graph has an empty vertex cover, and a graph with a single edge has an smallest vertex cover of size 1. Therefore, \( \tau(P_1) = \lceil (1-1)/2 \rceil = 0 \) and \( \tau(P_2) = \lceil (2-1)/2 \rceil = 1 \), as required.

To prove that the lemma holds for \( k \geq 3 \), assume it holds for all \( k' \) with \( 1 \leq k' < k \). Denote the sequence of vertices of the path \( P_k \) by \((v_1, v_2, \ldots, v_k)\). To cover the edge \( v_{k-1}v_k \), a vertex cover \( C \) needs to include \( v_{k-1} \) or \( v_k \) (or both). If \( v_k \in C \), then \( C' = (C \setminus \{v_k\}) \cup \{v_{k-1}\} \) is a vertex cover as well, and \( |C'| \leq |C| \). We conclude that there is a smallest vertex cover containing \( v_{k-1} \). The remaining vertices of the vertex cover need to cover the edges of the path \( P_{k-2} = (v_1, v_2, \ldots, v_{k-2}) \). Therefore,

\[
\tau(P_k) = 1 + \tau(P_{k-2}) = 1 + \lceil (k-3)/2 \rceil = \lceil (k-1)/2 \rceil.
\]

This concludes the proof of the lemma.

Lemma 4. For a cycle \( C_k \) on \( k \geq 3 \) vertices, \( \tau(C_k) = \lceil k/2 \rceil \).

Proof. Since \( C_k = (v_1, v_2, \ldots, v_k, v_1) \) has edges, its smallest vertex cover contains at least one vertex. By symmetry, there is a smallest vertex cover containing the vertex \( v_k \). The remaining vertices of the vertex cover need to cover the vertices of the path \( P_{k-1} = (v_1, v_2, \ldots, v_{k-1}) \). Thus,

\[
\tau(C_k) = 1 + \tau(P_{k-1}) = 1 + \lceil (k-2)/2 \rceil = \lceil k/2 \rceil.
\]

This concludes the proof of the lemma.

\(^1\)see the Glossary if these terms are unclear
To solve Vertex Cover on a graph of maximum degree at most 2, we compute its connected components (for example, by breadth-first search). For each connected component, we determine whether it is a path or a cycle, compute their number of vertices, and sum their vertex cover numbers using the previous lemmas. Each of these steps takes polynomial time.

**Exercise 3.** A vertex cover $C$ of a graph $G$ is minimal if no strict subset of $C$ is a vertex cover. Show that each graph has at most $2^k$ minimal vertex covers of size at most $k$. Furthermore, show that given $G$ and $k$, all minimal vertex covers of $G$ of size at most $k$ can be enumerated in $2^k n^{O(1)}$ time.

**Solution sketch.** Write a procedure to check whether a vertex cover is minimal in polynomial time; adapt Algorithm $vc1$ to enumerate all minimal vertex covers of size at most $k$ in $2^k n^{O(1)}$ time.