Utility theory

- Preference relations
- Consistent preference
- Preferences to values
- Evaluating prizes
- Evaluating lotteries
Utility theory

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- Consistent preference
- Preferences to values
- Evaluating prizes
- Evaluating lotteries

Example (Bus or train?)

Would Alice prefer to catch the bus or the train if:

- she’s a doctor on an emergency call
- has an injured foot
- is a tourist

- How to compare outcomes: travel time, walking distance, scenic appeal, comfort, etc.?
- How do we measure/quantify scenic appeal, comfort?
Preference

- Based preference on numerical values assigned to outcomes and actions: *i.e.*, prefer:
  - outcome $\omega_1$ to $\omega_2$ if $v(\omega_1) > v(\omega_2)$
  - action A to B if $V(A) > V(B)$
- Which value? *e.g.*, Alice is a tourist who values comfort and good scenery
- Does value determine preference or preference determine value?
- Can rational decisions be made when numerical values aren’t given/available?
- Are there alternatives to Bayes values?

Preference vs values

- Numbers aren’t always required; consider the *Maximin* rule:

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$v_{11}$</td>
<td>$v_{12}$</td>
</tr>
<tr>
<td>B</td>
<td>$v_{21}$</td>
<td>$v_{22}$</td>
</tr>
</tbody>
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<tr>
<th></th>
<th>$s_1$</th>
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<tbody>
<tr>
<td>A</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>16</td>
<td>8</td>
</tr>
</tbody>
</table>

- *Maximin* is independent of specific values assigned to outcomes, provided *preference order* is preserved: *i.e.*, $v_{11} > v_{21} > v_{22} > v_{12}$

Exercise

Will this be still be the case for *Hurwicz’s rule* ($\alpha = \frac{1}{4}$)? *miniMax Regret*? Laplace’s rule?
Utility theory  Preference relations

Qualitative preference: preference without numbers

- *Maximin* can be reformulated in terms of *qualitative preferences* only

<table>
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<th>$s_1$</th>
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<th>Preferences</th>
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<tbody>
<tr>
<td>$\omega_{11}$</td>
<td>$\omega_{12}$</td>
<td>$\omega_{11}$ preferred to $\omega_{21}$</td>
<td></td>
</tr>
<tr>
<td>$\omega_{21}$</td>
<td>$\omega_{22}$</td>
<td>$\omega_{21}$ preferred to $\omega_{22}$</td>
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<tr>
<td>$\omega_{22}$</td>
<td>$\omega_{12}$</td>
<td>$\omega_{22}$ preferred to $\omega_{12}$</td>
<td></td>
</tr>
</tbody>
</table>

**Definition (Qualitative *Maximin*)**

Associate an action with its/a least preferred outcome. Choose action whose associated outcome is most preferred.

- Which is least preferred outcome of A? *i.e.*, $\omega_{11}$ preferred to $\omega_{12}$?

Preference and value

Consequences of assigning numerical quantities (*i.e.*, via some value function $v : \Omega \rightarrow \mathbb{R}$) to encode preference:

- either prefer $a$ to $b$, or $b$ to $a$, or prefer them equally; *i.e.*, *indifferent* between $a$ and $b$
- if prefer $a$ to $b$, and $b$ to $c$, then prefer $a$ to $c$; *i.e.*, preferences *transitive*

**Questions**

- Are these conditions justified in practice?
- Do actual (human) agents always behave in this way?
- Can you find counter-examples?
Consistent preferences

- Rational decisions can be made without numerical values so long as an agent’s preferences are ‘consistent’
- What does ‘preference consistency’ mean?

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</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\omega_{22}$ preferred to $\omega_{12}$</td>
</tr>
</tbody>
</table>

- Examples:
  - $\omega_{11}$ preferred to $\omega_{12}$
  - $\omega_{21}$ not preferred to $\omega_{11}$

Preference consistency

- Rational (strict) preferences should be consistent in the sense that, e.g.:
  - if prefer apples (A) to bananas (B), then shouldn’t prefer bananas to apples
  - if prefer apples (A) to bananas (B) and bananas (B) to carrots (C), then shouldn’t prefer carrots (C) to apples (A)

Exercises

- What would be consequences of the failure of the first property above?
- In the second property above, should the agent then necessarily prefer apples to carrots?

- Preference is a binary relation
Utility theory  Preference relations

### Binary relations: overview

Modelling binary relations:
- If $A$ and $B$ are sets, define the *Cartesian product* of $A$ and $B$:
  \[ A \times B = \{(a, b) \mid a \in A \; \& \; b \in B\}; \text{ e.g., the set of all coordinate pairs on the Euclidean plane } \mathbb{R} \times \mathbb{R} \]

#### Definition (Binary relation)

A *binary relation* $R$ from $A$ to $B$ is a subset of $A \times B$; *i.e.*, $R \subseteq A \times B$.
Each ordered pair $(x, y) \in R$ is called an *instance* of $R$.

- In *infix notation*: $aRb$ iff $(a, b) \in R$; *e.g.*, $3 \leq 5$
- If $aRb$ (*i.e.*, $(x, y) \in R$) then the relation $R$ is said to *hold* for $x$ with $y$; *e.g.*, because $3 \leq 5$, then $\leq$ holds for $3$ with $5$

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Utility theory  Preference relations

### Binary relations

#### Definition (Binary relation on a set $A$)

A binary relation, $R$, *on* a set $A$ is a subset of $A \times A$; *i.e.*, $R \subseteq A \times A$.

- *e.g.*, the binary relation ‘is greater than’, written $> \subseteq \mathbb{R} \times \mathbb{R}$, is a binary relation on the set of real numbers $\mathbb{R}$ (and on $\mathbb{N}$, and on $\mathbb{Q}$)
- *e.g.*, the ‘greater than’ relation ($>$) holds between real numbers $3$ and $\pi$ (written $3 > \pi$); *i.e.*, $3 > \pi$ is an instance of $>$
Representing relations

- Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$. Relation $R \subseteq A \times B$ represented as matrix/table:

<table>
<thead>
<tr>
<th></th>
<th>$b_1$</th>
<th>$b_2$</th>
<th>$b_3$</th>
<th>$b_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$\times$</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_2$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$a_3$</td>
<td>$\times$</td>
<td></td>
<td></td>
<td>$\times$</td>
</tr>
</tbody>
</table>

- An $\times$ entry in row $x$ and column $y$ iff $xRy$. e.g., above $a_1Rb_2$, $a_2Rb_1$, and $a_3Rb_4$, but $a_1Rb_1$.

Relational properties

Let $R$ be a binary relation on some set $A$:

- $R$ is reflexive iff for every $x \in A$, $xRx$; e.g., for every $x \in \mathbb{R}$, $x = x$, $x \leq x$, $x \geq x$
- $R$ is irreflexive iff for every $x \in A$, $xRx$ does not hold; e.g., for every $x \in \mathbb{R}$, $x \neq x$, $x < x$, $x > x$ do not hold
- $R$ is transitive iff for any $x, y, z \in A$, when $xRy$ and $yRz$, then $xRz$; e.g., $=, <, \leq$ on $\mathbb{R}$
- $R$ is symmetric iff for any $x, y \in A$, when $xRy$, then $yRx$; e.g., $=, \leq$ on $\mathbb{R}$
- $R$ is total iff $xRy$ or $yRx$; e.g., $=, \leq$ on $\mathbb{R}$
- $R$ is asymmetric iff whenever $xRy$ then $yRx$ does not hold; e.g., $<$ on $\mathbb{R}$
- $R$ is antisymmetric iff whenever $xRy$ and $yRx$, then $x = y$; e.g., $\leq$ on $\mathbb{R}$
Preference relations

- A binary relation can be used to model strict preference:

**Definition**

An agent strictly prefers element $a$ to $b$, written $a \succ b$, iff it prefers $a$ more than $b$; *i.e.*, it would eliminate $b$. The collection of all such instances comprises the agent’s strict preference relation, $\succ$.

- What intuitive properties should strict preference relations satisfy?
  - if $x \succ y$, then it should not be the case that $y \succ x$
  - if $x \succ y$ and $y \succ z$, then it should not be the case that $z \succ x$

---

Representing preference: Hasse diagrams

- Given following strict preferences on $A = \{a, b, c, d, e, f, g\}$:

\[
\begin{align*}
  a & \succ g \\
  f & \succ c \\
  d & \succ a \\
  e & \succ b \\
  c & \succ a \\
  g & \succ b
\end{align*}
\]

- Do we know that $a \succ b$? What about $c \succ d$, $f \succ d$, $e \succ g$?
- $x \succ y$ iff there’s a path following arrows from $x$ to $y$
### Indifference and weak preference

#### Definition (Indifference)

If two elements $a$ and $b$ are *equally preferred* then the agent is said to be *indifferent* between them, written $a \sim b$. The set of all such instances constitutes an agent’s binary relation of *indifference*. The *indifference class* of $a$ is $[a] = \{b \mid a \sim b\}$.

#### Definition (Weak preference)

Element $a$ is *weakly preferred* to $b$, written $a \preceq b$, iff $a$ is strictly preferred to $b$ or the two are equally preferred; *i.e.*, $a$ is at least as preferred as $b$; *i.e.*, $a \preceq b$ iff $a \succ b$ or $a \sim b$.

### Indifference properties

The following are intuitive properties of indifference:

- if $x \sim y$, then $y \sim x$
- if $x \sim y$ and $y \sim z$, then $x \sim z$
- $x \sim x$ holds for any $x \in A$

Combined properties:

- if $x \sim y$ and $z \succ x$, then $z \succ y$
- if $x \sim y$ and $x \succ z$, then $y \succ z$

Note that, in the previous problem, it would be *inconsistent* for $c \sim d$ and $f \sim d$, as $f \succ c$, which would imply $f \succ d$. 
Consistent preference

- What does it mean for preferences to be consistent?
- Regard $\succeq$ as primitive; interpretation: $x \succeq y$ if “$x$ is at least as preferred as $y$”
- The following axioms characterise consistent preference

**Axiom 1: Transitivity**

The relation $\succeq$ is transitive; i.e., preference accumulates.

**Axiom 2: Comparability**

The relation $\succeq$ is total; i.e., every outcome is comparable.

**Derived definitions**

From $\succeq$ define indifference and strict preference:

**Definition (Indifference)**

The relation of indifference, denoted $\sim$, is defined by:

$$x \sim y \iff x \succeq y \& y \succeq x.$$  

**Definition (Strict preference)**

The relation of strict preference, denoted $\succ$, is defined by:

$$x \succ y \iff y \succeq x \text{ does not hold.}$$
Preference relations

Properties

The following properties follow from the earlier definitions:

- Indifference $\sim$ is an equivalence relation
- The corresponding strict preference relation $\succ$ is a strict total order
- Strict preference satisfies an indifference version of the trichotomy law
  i.e., exactly one of the following holds for any $x, y \in A$: $x \succ y$ or $x \sim y$ or $y \succ x$.

Values from preferences

Definition (Ordinal value function)

An ordinal value function on a ‘preference set’ $(A, \succeq)$ is a function $v : A \rightarrow \mathbb{R}$ such that $v(x) \geq v(y)$ iff $x \succeq y$.

Exercise

Show that for any ordinal value function $v$:

- $v(x) > v(y)$ iff $x \succ y$
- $v(x) = v(y)$ iff $x \sim y$

Theorem (Consistency)

For any consistent preference relation there exists an ordinal value function.
Values from weak preference

Consider a complete list of weak preferences on a set \( A = \{a, b, c, d\} \):

\[
\begin{align*}
  a &\succeq c & c &\succeq a & b &\succeq d & d &\succeq a & d &\succeq c \\
  b &\succeq a & b &\succeq c \\
  a &\succeq c & c &\succeq a & b &\succeq d & d &\succeq a & d &\succeq c \\
  a &\sim c & b &\succ d & d &\succ a & d &\succ c
\end{align*}
\]

\[
\begin{array}{c}
  b \\
  \downarrow \\
  \succeq \\
  \downarrow \\
  \sim \\
  \downarrow \\
  \succ \\
  \downarrow \\
  \succeq
\end{array}
\quad
\begin{array}{c}
  a \\
  \downarrow \\
  \succeq \\
  \downarrow \\
  \sim \\
  \downarrow \\
  \succ \\
  \downarrow \\
  \succeq
\end{array}
\]

\[
\begin{array}{c}
  b \\
  \downarrow \\
  d \\
  \rightarrow \\
  a \\
  \downarrow \\
  c
\end{array}
\]

\[
\begin{array}{c}
  b \\
  \downarrow \\
  d \\
  \rightarrow \\
  a \\
  \downarrow \\
  c
\end{array}
\]

The rank of \( x \) is \( r(x) = \) number of \( \times \) in \( x \)'s row; e.g.,
\[
  r(b) = 2, r(d) = 1, \text{ and } r(a) = r(c) = 0.
\]

This ranking is an ordinal value function.
Utility theory  Preferences to values

**Ordinal ranking**

![Diagram showing ordinal ranking]

**Definition (Rank)**

The *rank* of an indifference is defined by the successive values assigned to indifference class when the lowest indifference class is assigned rank 0.

i.e., the ranks above are 0, 1, 2, . . .

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**Evaluating prizes**

- Suppose the prizes in lottery $\ell$ ordered by preference: $a \succ b \succ c \succ d$.
- Choose reference values for best and worst prizes, $a$ and $d$: e.g., $v(a) = 100$ and $v(d) = 0$

\[
\begin{array}{c}
0 \\
d \\
c \\
b \\
a \\
\end{array} \quad \begin{array}{c}
75 \\
100 \\
v \\
\end{array}
\]

- Which value should be assigned to $b$? $100 \times \text{rank}(b)/\text{rank}(a)$?
- Assume: $b \sim [\frac{3}{4} : a][\frac{1}{2} : d]$
- Assign $v(b)$ to be proportional to $p$ (i.e., $\frac{3}{4}$); i.e., $v(b) = \frac{3}{4} \times 100 = 75$
Consistent preference

Axiom: continuity
If \( a \succeq b \succeq c \) then there is some \( p \in [0, 1] \), such that:
\[
 b \sim [p : a | (1 - p) : c]
\]
Interpretation: every intermediate prize is preferred equally to some lottery of the two extremal prizes.

Axiom: monotonicity
If \( A \succeq B \), then:
\[
 [p : A | (1 - p) : B] \succeq_L [p' : A | (1 - p') : B] \quad \text{iff} \quad p \geq p'.
\]
Interpretation: when the prizes in two lotteries are the same, the lottery which gives a better chance of the more preferred prize should be preferred; \( i.e. \), \( p \) is a measure of preference over same prizes.

Evaluating prizes

- Suppose the prizes in lottery \( \ell \) ordered by preference: \( a \succ b \succ c \succ d \).
- Choose reference values for best and worst prizes, \( a \) and \( d \): \( e.g. \), \( v(a) = 100 \) and \( v(d) = 0 \)

\[
 v(b) = \frac{3}{4} \times 100 = 75
\]

- Which value should be assigned to \( b \)? \( 100 \times \text{rank}(b) / \text{rank}(a) \)?
- Assume: \( b \sim [\frac{3}{4} : a | \frac{1}{4} : d] \)
- Assign \( v(b) \) to be proportional to \( p \) (\( i.e., \), \( \frac{3}{4} \)) \( i.e., \), \( v(b) = \frac{3}{4} \times 100 = 75 \)
Consistent preference

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If \( a \succeq b \succeq c \) then there is some \( p \in [0, 1] \), such that:
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If \( A \succeq B \), then:
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\]
Interpretation: when the prizes in two lotteries are the same, the lottery which gives a better chance of the more preferred prize should be preferred; i.e., \( p \) is a measure of preference over same prizes.

Evaluating intermediate prizes

\[
\frac{v(x) - v(d)}{v(a) - v(d)} = p_x
\]
where \( 0 \leq p_x \leq 1 \), assign value \( v(x) \), where:
\[
\frac{v(x) - v(d)}{v(a) - v(d)} = p_x
\]
i.e., \( v(x) = \alpha p_x + \beta \), where \( \alpha = v(a) - v(d) \) and \( \beta = v(d) \)
Binary lotteries

Definition (Binary lottery)

A binary lottery is a lottery in which at most two possible prizes have non-zero probability: i.e., of the form \( \ell = [p : A | (1 - p) : B] \).

\[
p \quad \bullet \quad A
\]
\[
1 - p \quad \bullet \quad B
\]

E.g., the lottery for tossing a fair coin: \( \ell = [\frac{1}{2} : h | \frac{1}{2} : t] \).

Reference lotteries

Definition (Reference lottery)

Let \( \omega_M \) and \( \omega_m \) be, respectively, the best and worst possible prizes (\( \omega_M \succ \omega_m \)). A reference lottery, \( \ell^* \), is a binary lottery:

\[
\ell^* = [p : \omega_M | (1 - p) : \omega_m]
\]

If prize \( x \sim \ell_x^* = [p_x^* : \omega_M | (1 - p_x^*) : \omega_m] \), then \( \ell_x^* \) is called the reference lottery for \( x \), and \( p_x^* \) is called the reference probability of \( x \).
Utility of a prize

**Definition (Utility of a prize)**

Define function $u : \Omega \rightarrow \mathbb{R}$, such that if $\omega \sim \ell^*_\omega = [p^*_\omega : \omega_M | (1 - p^*_\omega) : \omega_m]$, then $u(\omega) = E_u(\ell^*_\omega)$ (where $0 \leq p^*_\omega \leq 1$).

- Interpretation: the utility of a prize is proportional to the reference probability of the prize; specifically:
  
  if $u(\omega_m) = 0$ and $u(\omega_M) = 1$, then $u(\omega) = p^*$

- In general:
  
  $u(\omega) = p^*_\omega (v(\omega_M) - v(\omega_m)) + v(\omega_m)$

Preferences over lotteries

- Decisions typically involve preference over lotteries/actions
- Define preference over lotteries, $\succsim_L$

**Definition (Lottery preference)**

For lotteries $\ell$ and $\ell'$, we write $\ell \succsim_L \ell'$ iff $\ell$ is at least as preferred as $\ell'$.

**Definition (Inductive definition of lotteries)**

For any $n \in \mathbb{N}$, and $p_1, \ldots, p_n$, where $0 \leq p_i \leq 1$ and $\sum_i p_i = 1$:

- if $\omega \in \Omega$ is a prize, then $[\omega]$ is a lottery
- if $\ell_1, \ldots, \ell_n$ are lotteries, then $[p_1 : \ell_1 | \ldots | p_n : \ell_n]$ is a lottery

- Note: lotteries in general may have other lotteries as ‘prizes’
Composite lotteries

Lotteries may have other lotteries as prizes; e.g.,

\[ \ell = [p : A | 1 - p : [q : B | 1 - q : C]] \]

\[
\begin{array}{c}
\text{\(\ell\)} \\
\text{\(A\)} \\
\text{\(B\)} \\
\text{\(\ell'\)} \\
\text{\(C\)}
\end{array}
\]

\[
\begin{array}{c}
p \cdot A \\
1 - p \\
q \cdot B \\
1 - q \cdot C \\
(1 - p)(1 - q) \cdot C
\end{array}
\]

Agents should be indifferent between similar lotteries; e.g., \(\ell \sim_L \ell'\) above.

Composite lotteries: combination

Repeated outcomes can be combined/merged; e.g.,

\[
\begin{array}{c}
p \cdot A \\
1 - p \\
q \cdot A \\
1 - q \cdot C \\
(1 - p)(1 - q) \cdot C
\end{array}
\]

\[
\begin{array}{c}
p + (1 - p)q \cdot A \\
(1 - p)(1 - q) \cdot C
\end{array}
\]

These two should be equivalent:

\[
\]
**Reduction of composite lotteries**

**Axiom: substitution of equivalents**

If \( \ell \sim \ell' \), then any substitution of one for the other in a composite lottery will yield lotteries that equally preferred.

**Definition (Simple and composite lotteries)**

A *composite lottery* is one for which at least one prize is itself a lottery. A lottery which is not composite is said to be *simple* (or *flat*).

**Theorem: lottery reduction**

Composite lotteries can be reduced to equivalent (in regard to indifference) simple lotteries by combining probabilities in the usual way.

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**Normal lottery form**

Suppose \( A_n \succeq A_{n-1} \succeq \cdots \succeq A_1 \), with \( A_n \succ A_1 \).

In lottery \( \ell = [p_1 : A_1 | p_2 : A_2 | \cdots | p_n : A_n] \), replace \( A_i \) with \( [p_{A_i}^* : A_n | (1 - p_{A_i}^*) : A_1] \).
Standard lottery reduction

The lottery on the left can be combined to:

\[ p = p_1p^*_A + p_2p^*_A + \cdots + p_np^*_A. \]

Since \( p^*_A = u(A) \), this gives:

\[ p = p_1u(A) + \cdots + p_nu(A_n). \]

Utility theory

Axioms

- **consistent preferences**: extended to lotteries
- **monotonicity**: between binary lotteries
- **substitution of equivalents**
- **reduction of composite lotteries**: by flattening, merging outcomes, and combining probabilities
- **continuity**: each outcome has an equivalent binary (standard) lottery

Theorem (Utility existence)

*If the above axioms are satisfied, then there exists a linear function \( u : \Omega \to \mathbb{R} \) such that \( \omega_1 \succeq \omega_2 \) iff \( u(\omega_1) \geq u(\omega_2) \). Moreover, each \( u \) can be extended to a linear function \( U \) over lotteries, such that \( \ell \succeq \ell' \) iff \( U(\ell) \geq U(\ell') \), where \( U(\ell) = V_B(\ell) = E(u) \).*
The Maximal Utility Principle

Proof

By continuity assign \( u(\omega) = p_\omega^* \) from \( \omega \)’s equivalent reference lottery \( \ell_\omega^* \). Reduce each lottery \( \ell \) to its equivalent reference lottery \([p_\ell : \omega_M](1 - p_\ell) : \omega_m]\). Moreover, by monotonicity \( \ell \succeq \ell' \) iff \( p_\ell \geq p_{\ell'} \); i.e., iff \( p_1u(A_1) + \cdots + p_nu(A_n) \geq p'_1u(A_1) + \cdots + p'_nu(A_n) \). But these are just \( E_p(u) \geq E_{p'}(u) \). For lottery \( \ell \) set:

\[
U(\ell) = V_B(\ell) = E(u) = p_1u(A_1) + \cdots + p_nu(A_n)
\]

Maximal Expected Utility Principle (MEUP)

Rational agents prefer lotteries with greater expected utility over the prizes.

The MEUP justifies the Bayes decision rule as the rational rule for decision problems involving risk.

Utility: summary

- Preference is the fundamental notion in evaluating outcomes and actions/strategies
- Preference is a binary relation over outcomes/strategies/lotteries
- Consistent preferences lead to well-defined ‘utilities’ with which measure/quantify our preferences
- Bayes rule is the rational decision rule for evaluating strategies under risk