Risk attitudes and Utility

1 Risk
   - Risk preference
   - Expected monetary value

2 Utility
   - Utility of money
   - Risk attitudes
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Introduction to risk preference

To gamble or not to gamble

You’re offered to play the following game: a coin is tossed once. If it lands ‘heads’ you get $2000. If it lands ‘tails’ you get nothing. It costs $1000 to play. Would you play?

- Measured in dollars, \( v_s(x) = x \), the two have equal Bayes value; i.e., \( v_s($1000) = 1000 = V_B([\frac{1}{2} : $2000|\frac{1}{2} : $0]) \)
- Most won’t risk $1000 on this bet; i.e., prefer $1000 to \([\frac{1}{2} : $2000|\frac{1}{2} : $0]\)
- How can we explain this?

Is this irrational?
Do we need a new decision rule (other than Bayes)?
Gambles

Definition (Gamble)
A *gamble* is a decision problem with two alternatives: one which is certain and another which is a (proper) lottery.

Examples
Whether or not to:
- bet on the toss of a coin
- bet on a horse race, a football match, *etc.*
- buy a share whose price may go up or down
- pay for insurance

Expected monetary value

Definition (Expected monetary value)
The *expected monetary value* (EMV) of a lottery, denoted $V_S$, is the *Bayes* value of the lottery when outcomes are valued in $\$ (i.e., $v = v_S$).

$$V_S(\ell_G) = v_S(\$x) = x$$
$$V_S(\ell_G) = \frac{1}{2} v_S(h) + \frac{1}{2} v_S(t) = \frac{1}{2} (2000) + \frac{1}{2} (0) = 1000$$

How much would you pay to gamble?
**Risk attitude indicators**

**Definition (Certainty equivalent)**
An agent’s *certainty equivalent* for a lottery is the certain amount it would be willing to exchange for the lottery; *i.e.*, the certain amount for which an agent would be indifferent between it and the lottery.

- certainty equivalents are subjective: different decision-makers may have different certainty equivalents for the same lottery
- certainty equivalents characterise risk attitudes towards a lottery: what would a ‘high’ certainty equivalent mean?

**Definition (Risk premium)**
An agent’s *risk premium* for a lottery is the difference between the lottery’s fair bet value (EMV) and the agent’s certainty equivalent.

**Fair gambles**

**Definition (Fair gamble)**
A gamble is *fair* or *unbiased* if the expected monetary value for the lottery is the same as the value of the certain outcome; *i.e.***,

\[
V_S(\ell_G) = E(v_S) = V_S(\ell_G)
\]

Suppose Alice has $10 and is offered a bet on $\begin{array}{c}
\frac{1}{5} : \$50 \\
\frac{4}{5} : \$0
\end{array}$.

Questions:
- is the bet fair?
- should she gamble?
Risk attitude indicators

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Should Alice bet if she *believes* the chances of winning exceed 1 in 5? Suppose she needs $10 to buy dinner; should Alice gamble?

Alice’s risk preference: I’ll gamble (risk going hungry) only if my chances of winning are at least even (*i.e.*, greater than 1 in 2); *i.e.*, indifferent between certain $10 and \( \ell = \left[ \frac{1}{2} : \$50 | \frac{1}{2} : \$0 \right] \):

\[
u(\$10) = U\left( \left[ \frac{1}{2} : \$50 | \frac{1}{2} : \$0 \right] \right) = E_u(\ell)
= V_B\left( \left[ \frac{1}{2} : \$50 | \frac{1}{2} : \$0 \right] \right) \quad \text{using } u \text{ rather than } v$
= \frac{1}{2}u(\$50) + \frac{1}{2}u(\$0)

What does \( u \) look like?

Reference scale for \( u \) relative to best/worst outcomes:

\[
u(\$0) = 0
\]
\[
u(\$50) = 1
\]

Reference lotteries lie on diagonal:

\[
U\left( \left[ \frac{1}{2} : \$50 | \frac{1}{2} : \$0 \right] \right) = \frac{1}{2}u(\$50) + \frac{1}{2}u(\$0) = \frac{1}{2}
\]
\[
U\left( \left[ p : \$50 | (1 - p) : \$0 \right] \right) = p
\]
Utility of money

On Alice’s utility scale the monetary outcomes are arranged as follows:

<table>
<thead>
<tr>
<th>$0</th>
<th>$0.5</th>
<th>$1</th>
<th>$2</th>
<th>$5</th>
<th>$10</th>
<th>$25</th>
<th>$50</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0</td>
<td>$3</td>
<td>$10</td>
<td>$25</td>
<td>$50</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Question**
What properties do typical utility functions for money have?

**Utility values should increase with increasing money**

Functions on ordered sets

**Definition (Monotonic increasing function)**

A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **monotonically increasing**, or **non-decreasing**, iff for any $x, y \in \mathbb{R}$, if $x \geq y$, then $f(x) \geq f(y)$.

Examples: the following are non-decreasing functions on $\mathbb{R}$: $f(x) = \frac{1}{10} x$, $f(x) = x$, $f(x) = c$, for any fixed $c \in \mathbb{R}$

**Exercise**

Does this imply the converse; i.e., if $f(x) \geq f(y)$, then $x \geq y$?
Utility

Strictly increasing functions

Definition (Strictly increasing function)
A real-valued function \( f : \mathbb{R} \to \mathbb{R} \) is strictly increasing iff for any \( x, y \in \mathbb{R} \), if \( x > y \), then \( f(x) > f(y) \).

Examples: \( f(x) = \frac{1}{10} x \), \( f(x) = x \), \( f(x) = 3x + 2 \), \( f(x) = x^2 \) for \( x \geq 0 \), \( f(x) = \log_2 x \)

Utility for money

Definition (Certainty equivalent)
An agent’s certainty equivalent for a lottery is the value \( x_c \) for which the agent would be indifferent between it and the lottery; i.e., \( u(x_c) = U(\ell) \).

Definition (Risk premium)
The risk premium of an agent for lottery \( \ell \) is the difference between the EMV of the lottery and the certainty equivalent: \( V_S(\ell) - x_c \).
Repeated trials

Example (Alice and Bob)

Alice and her twin, Bob, have $10 each and they are offered, separately, 4 to 1 odds on a team in two different football matches (e.g., home and away). They believe the team has a 2 in 5 chance of winning each match.

Should Alice bet?

Outcomes if both gamble:

\[ \ell_{AB} = \left[ \frac{9}{25} : (0, 0) | \frac{6}{25} : (0, 50) | \frac{6}{25} : (50, 0) | \frac{4}{25} : (50, 50) \right] \]

If Alice and Bob share the risk/gain then:

\( (x, y) \sim \$ \left( \frac{x+y}{2} \right) \)

i.e. \( u_A(x, y) = u_A \left( \frac{x+y}{2} \right) \)

So for Alice:

\[ \ell_A = \left[ \frac{9}{25} : 0 | \frac{6}{25} : 25 | \frac{6}{25} : 25 | \frac{4}{25} : 50 \right] \]

\[ = \left[ \frac{9}{25} : 0 | \frac{12}{25} : 25 | \frac{4}{25} : 50 \right] \]

Where does \( \ell_A \) fit in in the scheme of things?

\[ \ell_A = \left[ \frac{9}{25} : 0 | \frac{12}{25} : 25 | \frac{4}{25} : 50 \right] \]

\[ V_S(\ell_A) = \frac{12}{25} (25) + \frac{4}{25} (50) = 20 \]

\[ U_A(\ell_A) = \frac{9}{25} (0) + \frac{12}{25} u_A(25) + \frac{4}{25} (1) \]

\[ \approx 0 + \frac{12}{25} \left( \frac{9}{10} \right) + \frac{4}{25} = \frac{4}{25} \left( \frac{37}{10} \right) \]

\[ > \frac{4}{25} \left( \frac{35}{10} \right) = \frac{14}{25} > \frac{1}{2} = u_A(10) \]

Alice should bet, sharing the risk and the winnings!
Repeated trials

- The individual bets are favourable for both Alice and Bob
- Despite this neither Alice nor Bob would take their respective individual bets
- However, they should bet together over multiple bets/trials

Risk attitudes

Definition (Risk attitudes)

An agent is:

- **risk averse** iff its certainty equivalent is less than the lottery’s expected value; *i.e.*, it values the lottery to be worth less than the expected value.
- **risk seeking** (**risk prone**) iff its certainty equivalent is greater than the lottery’s expected value.
- **risk-neutral** otherwise.

Exercises

- What is Alice’s certainty equivalent for the lottery with Bob?
- The risk premium in what range if the agent is: risk averse? risk seeking? risk neutral?
Risk attitudes

For a lottery:

**Definition (Risk averse)**

An agent is *risk averse* if its utility function is concave down over the range of possible outcomes.

**Definition (Risk seeking)**

An agent is *risk seeking* if its utility function is concave up (convex) over the range of possible outcomes.

**Definition (Risk neutral)**

An agent is *risk neutral* if its utility function both concave down and up; i.e., linear.

Concave and convex functions

**Definition (Concave and convex)**

A function $f : \mathbb{R} \to \mathbb{R}$ is *concave down* in the interval $[a, b]$ if for all $x, y \in [a, b]$, and all $\lambda \in [0, 1]$, $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$, and *concave up* (or convex) if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.

where $\nu = \lambda x + (1 - \lambda)y$
Summary: risk attitudes and utility

- Not all quantities (e.g., $) accurately represent ‘true’ preference
- Measure preference in terms of utility; agent must calibrate utilities against uncertain outcomes (lotteries)
- An agent’s utility is personal/subjective; i.e., particular to him. Different agents may have different utilities for the same ‘outcome’
- Utility functions are non-decreasing; this means that over many trials Bayes utilities approach expected values
- The shape of an agent’s utility curve/function determines its risk attitude